SOLUTION 1. In Section 5.3, it is shown that the power spectral density is

\[ S_X(f) = \frac{|\psi_F(f)|^2}{T} \sum_k K_X[k] \exp(-j2\pi kfT), \]

where \( K_X[k] \) is the auto-covariance of \( X \), and \( \psi_F(f) \) is the Fourier transform of \( \psi(t) \). Because \( \{X_i\}_{i=-\infty}^{\infty} \) are i.i.d. and have zero-mean,

\[ K_X[k] = \mathbb{E}[X_{i+k}X_i^*] = \mathbb{E}\{k = 0\}, \]

so

\[ S_X(f) = \mathbb{E}\left|\frac{\psi_F(f)}{T}\right|^2. \]

Moreover,

\begin{align*}
\psi_F(f) &= \int_{-\infty}^{\infty} \psi(t)e^{-j2\pi ft} \, dt \\
&= \frac{1}{\sqrt{T}} \int_{0}^{\frac{T}{2}} e^{-j2\pi ft} \, dt - \frac{1}{\sqrt{T}} \int_{\frac{T}{2}}^{T} e^{-j2\pi ft} \, dt \\
&= \frac{j}{2\pi f\sqrt{T}} \left( e^{-j2\pi f\frac{T}{2}} - 1 - e^{-j2\pi fT} + e^{-j2\pi f\frac{T}{2}} \right) \\
&= \frac{j}{2\pi f\sqrt{T}} e^{-j2\pi f\frac{T}{2}} \left( 2 - e^{j2\pi f\frac{T}{2}} - e^{-j2\pi f\frac{T}{2}} \right) \\
&= \frac{j}{2\pi f\sqrt{T}} e^{-j2\pi f\frac{T}{2}} \left( 2 - 2\cos(\pi fT) \right) \\
&= \frac{j}{2\pi f\sqrt{T}} e^{-j2\pi f\frac{T}{2}} 4\sin^2 \left( \pi f\frac{T}{2} \right). \end{align*}

Therefore,

\[ S_X(f) = \mathbb{E}\left|\frac{\psi_F(f)}{T}\right|^2 = \mathbb{E}\left(\pi f\frac{T}{2}\right)^2 \text{sinc}^4 \left( \pi f\frac{T}{2} \right). \]

SOLUTION 2.

(a) When \( i = j \), \( \mathbb{E}[X_iX_j] \) equals

\[ \mathbb{E}[X_i^2] = \mathbb{E}[1] = 1. \]

Remember that the \( B_i \) are i.i.d. Bernoulli(\( \frac{1}{2} \)) random variables. Hence, we find immediately

\[ K_X[1] = \mathbb{E}[X_{2n}X_{2n+1}] = \mathbb{E}[B_nB_{n-2}B_nB_{n-1}B_{n-2}] \]
\[ = \mathbb{E}[B_n^2B_{n-1}B_{n-2}] \]
\[ = \mathbb{E}[B_{n-1}] = 0, \]
and also
\[ K_X[2] = \mathbb{E}[X_{2n}X_{2n+2}] = \mathbb{E}[B_nB_{n-2}B_{n+1}B_{n-1}] = \mathbb{E}[B_n]\mathbb{E}[B_{n-2}B_{n+1}]\mathbb{E}[B_{n-1}] = 0. \]

By continuing this argument we find
\[ K_X[i] = 1 \{ i = 0 \}. \]

Hence,
\[ S_X(f) = \frac{\mathcal{E}_s}{T_s} |\psi_f(f)|^2. \]

This means that by choosing \( \psi(t) \) appropriately, we can control the bandwidth consumption of our communications scheme.

(b) We know that
\[ |\psi_f(f)|^2 = T_s \text{sinc}^2(T_s f). \]

It follows that
\[ S_X(f) = \mathcal{E}_s \text{sinc}^2(T_s f). \]

A plot of \( S_X(f) \) is shown below:

\begin{center}
\includegraphics{plot.png}
\end{center}

**Solution 3.**

(a) \( X_i = B_i - 2B_{i-1} \)

From this, we can draw the following trellis:

\begin{center}
\includegraphics{trellis.png}
\end{center}

(b) We have \( Y = X + Z \), where \( Z = (Z_1, \ldots, Z_6) \) is a sequence of i.i.d. components with \( Z_i \sim \mathcal{N}(0, \sigma^2) \). Our maximum likelihood decoder is a minimum distance decoder. We have to minimize \( \|y - x\|^2 \) or equivalently, maximize \( 2\langle y, x \rangle - \|x\|^2 \). We thus have \( f(x, y) = \sum_{i=1}^6 (2y_i x_i - x_i^2) \) whose maximization with respect to \( x \) leads to a maximum likelihood decision on \( X \).
(c) We label our trellis with the edge metric $2y_i x_1 - x_1^2$ and then trace back the decoding path.

We see that the two sequences 1,1,0,0,0 and 1,1,0,1,1 are equally likely, so the decoder would choose either of the two.

**Solution 4.** The trellis representing the encoder is shown below:

We display the diagram labeled with edge-metric according to the received sequence and state-metric of the survivor path. We also indicate the survivor paths and the decoding path.

From the figure we can read the decoded sequence 1,1,−1,1,1.

**Solution 5.** The output of encoder (a) is

$$T(\bar{x}_{2j-1}) = T(\bar{b}_j + \bar{b}_{j-2}) = T(\bar{b}_j)T(\bar{b}_{j-2}) = b_j b_{j-2}$$
$$T(\bar{x}_{2j}) = T(\bar{b}_j + \bar{b}_{j-1} + \bar{b}_{j-2}) = T(\bar{b}_j)T(\bar{b}_{j-1})T(\bar{b}_{j-2}) = b_j b_{j-1} b_{j-2},$$

which is identical to the output of encoder (b).