3 problems, 5 points in total
30 minutes

Good Luck!

PLEASE WRITE YOUR NAME ON THIS PAGE.
Problem 1. Consider a binary hypothesis test with observation \( Y = (Y_1, \ldots, Y_n), \ n \geq 2 \). When \( H = 0 \), \( \{Y_i\} \) are independent Gaussian random variables with zero mean and unit variance. When \( H = 1 \), \( \{Y_i\} \) are independent Gaussian random variables with zero mean and variance 2. For each of the following functions, indicate if it is a sufficient statistic, and briefly explain why.

*Hint:* Simplify the log-likelihood ratio.

(a) (0.25 pts) \( T_1(Y) = \sum_i Y_i \) Yes ☐ No ☐

(b) (0.25 pts) \( T_2(Y) = \sum_i Y_i^2 \) Yes ☐ No ☐

(c) (0.25 pts) \( T_3(Y) = \sum_i |Y_i| \) Yes ☐ No ☐

(d) (0.25 pts) \( T_4(Y) = \max_i |Y_i| \) Yes ☐ No ☐
Problem 2. Consider an $m$-ary hypothesis testing problem with the observation $Y = (Y_1, Y_2) \in \mathbb{R}^2$. Under hypothesis $i$, $Y = c_i + Z$, where $Z = (Z_1, Z_2)$ is distributed according to $N(0, \sigma^2 I)$. Suppose that for each $i$, $c_i$ is of the form $(a_i, -a_i)$.

(a) (1 pt) Show that $\tilde{Y} = Y_1 - Y_2$ is a sufficient statistic.  
*Hint:* Use the Fisher-Neyman Factorization Theorem.

(b) (0.75 pts) Suppose $U \in \mathbb{R}$ is another observation given by $U = a_i + W$ when $H = i$, and $W$ is distributed according to $N(0, \sigma^2/3)$. We are given the choice to observe either $Y$ or $U$. Which one should we choose to minimize the error probability under the optimal decision rule? 
*Hint:* Observing $U$ or $2U$ has the same error probability. Compare observing $\tilde{Y}$ with observing $2U$. 

Problem 3. In a binary hypothesis testing problem with equally-likely hypotheses, the observation is $Y = 1 + Z$ if $H = 0$ and $Y = -1 + Z$ if $H = 1$. The noise $Z$ has a Laplace distribution, i.e., $f_Z(z) = (1/2) \exp(-|z|)$.

(a) (0.75 pts) Find the optimal decision rule.

Hint: Your final answer should not contain any absolute value.

(b) (0.75 pts) Find the probability of error of this rule.

(c) (0.75 pts) Compute the Bhattacharyya bound.

Hint: The Bhattacharyya upper bound is given by $P_e \leq \int_{\mathbb{R}} \sqrt{f_{Y|H}(y|0)f_{Y|H}(y|1)} \, dy$. 

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