SOLUTION 1.

(a) We have a binary hypothesis testing problem: The hypothesis $H$ is the answer you will select, and your decision will be based on the observation of $\hat{H}_L$ and $\hat{H}_R$. Let $H$ take value 1 if answer 1 is chosen, and value 2 if answer 2 is chosen. In this case, we can write the MAP decision rule as follows:

$$\Pr\{H = 1|\hat{H}_L = 1, \hat{H}_R = 2\} \quad \hat{H} = 1 \quad \hat{H} = 2$$

$$\Pr\{H = 2|\hat{H}_L = 1, \hat{H}_R = 2\}$$

From the problem setting we know the priors $\Pr\{H = 1\}$ and $\Pr\{H = 2\}$; we can also determine the conditional probabilities $\Pr\{\hat{H}_L = 1|H = 1\}$, $\Pr\{\hat{H}_L = 1|H = 2\}$, $\Pr\{\hat{H}_R = 2|H = 1\}$ and $\Pr\{\hat{H}_R = 2|H = 2\}$ (we have $\Pr\{\hat{H}_L = 1|H = 1\} = 0.9$ and $\Pr\{\hat{H}_L = 1|H = 2\} = 0.1$). Introducing these quantities and using the Bayes rule we can formulate the MAP decision rule as follows:

$$\frac{\Pr\{\hat{H}_L = 1, \hat{H}_R = 2|H = 1\} \Pr\{H = 1\}}{\Pr\{\hat{H}_L = 1, \hat{H}_R = 2\}} \quad \hat{H} = 1 \quad \hat{H} = 2$$

$$\frac{\Pr\{\hat{H}_L = 1, \hat{H}_R = 2|H = 2\} \Pr\{H = 2\}}{\Pr\{\hat{H}_L = 1, \hat{H}_R = 2\}}$$

Now, assuming that the event $\{\hat{H}_L = 1\}$ is independent of the event $\{\hat{H}_R = 2\}$ and simplifying the expression, we obtain

$$\Pr\{\hat{H}_L = 1|H = 1\} \Pr\{\hat{H}_R = 2|H = 1\} \Pr\{H = 1\} \quad \hat{H} = 1 \quad \hat{H} = 2$$

$$\Pr\{\hat{H}_L = 1|H = 2\} \Pr\{\hat{H}_R = 2|H = 2\} \Pr\{H = 2\},$$

which is our final decision rule.

(b) Evaluating the previous decision rule, we have

$$0.9 \times 0.3 \times 0.25 \quad \hat{H} = 1 \quad \hat{H} = 2$$

$$0.1 \times 0.7 \times 0.75,$$

which gives

$$0.0675 \quad \hat{H} = 1 \quad \hat{H} = 2$$

$$0.0525$$

This implies that the answer $\hat{H}$ is equal to 1.
SOLUTION 2.

(a) We can write the MAP decision rule in the following way:

\[
\frac{P_{Y|H}(y|1)}{P_{Y|H}(y|0)} \quad \frac{H=1}{H=0} \quad \frac{P_H(0)}{P_H(1)}
\]

Plugging in, we find

\[
\frac{\lambda_1^y e^{-\lambda_1}}{\lambda_0^y e^{-\lambda_0}} \quad \frac{H=1}{H=0} \quad \frac{p_0}{1 - p_0},
\]

and then

\[
\left( \frac{\lambda_1}{\lambda_0} \right)^y \quad \frac{H=1}{H=0} \quad \frac{p_0}{1 - p_0} e^{\lambda_1 - \lambda_0}
\]

Taking logarithms on both sides does not change the direction of the inequalities, therefore

\[
y \log \left( \frac{\lambda_1}{\lambda_0} \right) \quad \frac{H=1}{H=0} \quad \log \left( \frac{p_0}{1 - p_0} e^{\lambda_1 - \lambda_0} \right)
\]

Attention: the term \(\log(\lambda_1/\lambda_0)\) can be negative, and if it is, then dividing by it involves changing the direction of the inequality.

Suppose \(\lambda_1 > \lambda_0\). Then, \(\log(\lambda_1/\lambda_0) > 0\), and the decision rule becomes

\[
y \log \left( \frac{\lambda_1}{\lambda_0} \right) \quad \frac{H=1}{H=0} \quad \frac{p_0}{1 - p_0} e^{\lambda_1 - \lambda_0} \equiv \theta
\]

(b) We compute

\[
P_e(0) = \Pr\{Y > \theta|H = 0\} = \sum_{y=\lceil \theta \rceil}^{\infty} P_{Y|H}(y|0)
\]

\[
= 1 - \sum_{y=0}^{\lceil \theta \rceil} \frac{\lambda_0^y}{y!} e^{-\lambda_0},
\]

and by analogy

\[
P_e(1) = \Pr\{Y < \theta|H = 1\} = \sum_{y=0}^{\lfloor \theta \rfloor} P_{Y|H}(y|1)
\]

\[
= \sum_{y=0}^{\lfloor \theta \rfloor} \frac{\lambda_1^y}{y!} e^{-\lambda_1}
\]

Thus, the probability of error becomes

\[
P_e = p_0 \left( 1 - \sum_{y=0}^{\lceil \theta \rceil} \frac{\lambda_0^y}{y!} e^{-\lambda_0} \right) + (1 - p_0) \sum_{y=0}^{\lfloor \theta \rfloor} \frac{\lambda_1^y}{y!} e^{-\lambda_1}
\]
Now, suppose that \( \lambda_1 < \lambda_0 \). Then, \( \log(\lambda_1/\lambda_0) < 0 \), and we have to swap the inequality sign, thus

\[
y 
\begin{align*}
H \geq H^* \\
H = 1
\end{align*}
\]

\[
\log \left( \frac{\lambda_1}{\lambda_0} \right) \leq \theta
\]

The rest of the analysis goes along the same lines, and finally, we obtain

\[
P_e = p_0 \sum_{y=0}^{\lceil \theta \rceil} \frac{\lambda_0^y}{y!} e^{-\lambda_0} + (1 - p_0) \left( 1 - \sum_{y=0}^{\lceil \theta \rceil} \frac{\lambda_1^y}{y!} e^{-\lambda_1} \right)
\]

The case \( \lambda_0 = \lambda_1 \) yields \( \log(\lambda_1/\lambda_0) = 0 \), so the decision rule becomes \( \hat{H} = 0 \) \( \iff \hat{H} = 1 \), regardless of \( y \). Thus, we can exclude the case \( \lambda_0 = \lambda_1 \) from our discussion.

(c) Here, we are in the case \( \lambda_1 > \lambda_0 \), and we find \( \theta \approx 4.54 \). We thus evaluate

\[
P_e \approx \frac{1}{3} \left( 1 - \sum_{y=0}^{4} \frac{2^y}{y!} e^{-2} \right) + \frac{2}{3} \sum_{y=0}^{4} \left( \frac{10^y}{y!} e^{-10} \right) 
\]

\[
\approx 0.03705
\]

(d) We find \( \theta \approx 7.5163 \)

\[
P_e \approx \frac{1}{3} \left( 1 - \sum_{y=0}^{7} \frac{2^y}{y!} e^{-2} \right) + \frac{2}{3} \sum_{y=0}^{7} \left( \frac{21^y}{y!} e^{-20} \right) 
\]

\[
\approx 0.000885
\]

The two Poisson distributions are much better separated than in (c); therefore, it becomes considerably easier to distinguish them based on one single observation \( y \).

**Solution 3.** We use the Fisher–Neyman factorization theorem.

(a) Since \( Y \) is an i.i.d. sequence,

\[
P_{Y|H}(y|i) = \prod_{k=1}^{n} P_{Y_k|H}(y_k|i) = \lambda_i^{\sum_{k=1}^{n} y_k} e^{-n \lambda_i} \prod_{k=1}^{n} \frac{1}{(y_k)!} \]

\[
= e^{-n \lambda_i} \lambda_i^{\sum_{k=1}^{n} y_k} g_i(T(y)) \frac{1}{h(y)}
\]

(b) Since \( Z_1, \ldots, Z_n \) are i.i.d additive noise samples,

\[
f_{Y|H}(y|i) = \prod_{k=1}^{n} f_{Z_k|H}(y_k - \theta_i) = \prod_{k=1}^{n} \lambda_i e^{-\lambda_i(y_k - \theta_i)} 1 \{ y_k \geq \theta_i \}
\]

\[
= \lambda_i^n e^{n \lambda_i \theta_i} e^{-n \lambda_i \left( \frac{1}{n} \sum_{k=1}^{n} y_k \right)} g_i(T(y)) \min \{ y_1, \ldots, y_n \} \geq \theta_i
\]

with \( h(y) = 1 \).
(a) It is straightforward to check that $w_0(t)$ has unit norm, i.e., $\|w_0(t)\| = 1$, thus $\psi_1(t) = w_0(t)$. With $\psi_1(t)$ we can reproduce the first portion of $w_1(t)$ (for $t$ between 0 and 1). With $\psi_2(t)$ we need to be able to describe the remaining part of $w_1(t)$. Clearly $\psi_2(t)$ is as illustrated below. With $\psi_1(t)$ and $\psi_2(t)$ we also describe the part of $w_2(t)$ between $t = 0$ and $t = 2$. Hence $\psi_3(t)$ is selected as the unit-norm function that matches the part of $w_2(t)$ between $t = 2$ and $t = 3$. We immediately see that $w_3(t)$ is also a linear combination of $\psi_i(t)$, $i = 1, 2, 3$.

(b) Using the basis $\{\psi_1(t), \psi_2(t), \psi_3(t)\}$, one can give the following representation for the waveforms $w_i(t)$, $i = 0, \ldots, 3$:

$$w_0 = (1, 0, 0)^T, w_1 = (-1, 1, 0)^T, w_2 = (1, 1, 1)^T, w_3 = (1, 1, -1)^T$$

Solution 5.

(a) The optimal solution is to pass $R(t)$ through the matched filter $w(T - t)$ and sample the result at $t = T$ to get a sufficient statistic denoted by $Y$. (In this problem, $T = 1$.) Note that $Y = S + N$, where $S$ and $N$ are random variables denoting the signal and the noise components respectively. Under $H = i$, $Y \sim N(\alpha_i, N_0/2)$, where $\alpha_0, \ldots, \alpha_3$ are $3c$, $c$, $-c$ and $-3c$ respectively.

Let $\hat{X}$ be the recovered signal value at the receiver. Based on the nearest neighbor decision rule, the receiver chooses the value of $\hat{X}$ in the following fashion:

$$\hat{X} = \begin{cases} +3, & Y \in [2c, \infty) \\ +1, & Y \in [0, 2c) \\ -1, & Y \in [-2c, 0) \\ -3, & Y \in [-\infty, -2c) \end{cases} \quad (1)$$

(b) The probability of error is given by

$$P_e = \sum_{i=0}^{3} \frac{1}{4} \Pr\{\text{error}|H = i\}$$

$$= \frac{1}{4} \left[ Q \left( \frac{c}{\sqrt{N_0/2}} \right) + 2Q \left( \frac{c}{\sqrt{N_0/2}} \right) + 2Q \left( \frac{c}{\sqrt{N_0/2}} \right) + Q \left( \frac{c}{\sqrt{N_0/2}} \right) \right]$$

$$= \frac{3}{2} Q \left( \frac{c}{\sqrt{N_0/2}} \right)$$
(c) In this case under $H = i$, $Y \sim N(\alpha_i, N_0/2)$, where $\alpha_0, \ldots, \alpha_3$ are $\frac{9c}{4}, \frac{3c}{4}, -\frac{3c}{4}$ and $-\frac{9c}{4}$ respectively. Using the decision rule in (1), the probability of error is given by

$$P_e = \sum_{i=0}^{3} \frac{1}{4} \Pr\{\text{error} | H = i\}$$

$$= \frac{1}{4} \left[ Q \left( \frac{c/4}{\sqrt{N_0/2}} \right) + Q \left( \frac{5c/4}{\sqrt{N_0/2}} \right) + Q \left( \frac{3c/4}{\sqrt{N_0/2}} \right) + Q \left( \frac{c/4}{\sqrt{N_0/2}} \right) \right]$$

$$= \frac{1}{2} \left[ Q \left( \frac{c/4}{\sqrt{N_0/2}} \right) + Q \left( \frac{3c/4}{\sqrt{N_0/2}} \right) + Q \left( \frac{5c/4}{\sqrt{N_0/2}} \right) \right]$$

(d) The noise process $N(t)$ is a stationary Gaussian random process. So the noise component $N$ (which is the sample of match-filter output at time $T$) is a Gaussian random variable with mean

$$\mathbb{E}[N] = \mathbb{E} \left[ \int_{-\infty}^{\infty} N(t)w(t)dt \right] = \mathbb{E} \left[ \int_{0}^{1} N(t)dt \right] = 0$$

Because the process $N(t)$ is stationary, without loss of generality we choose the boundaries of the integral to be $0$ and $T$ where in this problem $T = 1$.

Now, let us calculate the noise variance.

$$\text{var}(N) = \mathbb{E}[N^2] - \mathbb{E}[N]^2 = \mathbb{E}[N^2]$$

$$= \mathbb{E} \left[ \int_{-\infty}^{\infty} N(t)w(t)dt \int_{-\infty}^{\infty} N(v)w(v)dv \right]$$

$$= \mathbb{E} \left[ \int_{0}^{1} N(t)dt \int_{0}^{1} N(v)dv \right]$$

$$= \mathbb{E} \left[ \int_{0}^{1} \int_{0}^{1} N(t)N(v)dt dv \right]$$

$$= \int_{0}^{1} \int_{0}^{1} K_N(t-v)dt dv$$

$$= \int_{0}^{1} \int_{0}^{1} \frac{1}{4\alpha} e^{-|t-v|/\alpha} dt dv$$

$$= \frac{1}{2} \left( \alpha (e^{-1/\alpha} - 1) + 1 \right)$$

Thus the new probability of error is given by

$$P_e = \sum_{i=0}^{3} \frac{1}{4} \Pr\{\text{error} | H = i\}$$

$$= \frac{1}{4} \left[ Q \left( \frac{c}{\sqrt{\text{var}(N)}} \right) + 2Q \left( \frac{c}{\sqrt{\text{var}(N)}} \right) + 2Q \left( \frac{c}{\sqrt{\text{var}(N)}} \right) + Q \left( \frac{c}{\sqrt{\text{var}(N)}} \right) \right]$$

$$= \frac{3}{2} Q \left( \frac{c}{\sqrt{\frac{1}{2} (\alpha (e^{-1/\alpha} - 1) + 1)}} \right)$$