Exercise 1 Construction of a multi-control-$U$.

Verify that the multi-control-$U$:

$$
|c_1\rangle -|c_1\rangle \\
|c_2\rangle -|c_2\rangle \\
|c_3\rangle -|c_3\rangle \\
|t\rangle -U^{c_1c_2c_3}|t\rangle
$$

can be realized with the Toffoli gate (control-control-NOT) and a simple control-$U$.

Exercise 2 Controlled-controlled-$U$

In the last exercise, we have seen the construction of multi-controlled-$U$ gate with the Toffoli gates and a simple controlled-$U$ gate. This time, we are going to see a different construction for controlled-controlled-$U$ gate.

Let $V$ be any quantum gate such that $V^2 = U$. Prove the following circuit identity.

$V^2 = U$

Explain in what sense this second construction is "not universal", whereas the one of the previous exercise is "universal" (for a given fixed $U$).
Exercise 3  Construction of the Toffoli gate from a control-NOT (hint : long calculation).

Verify that the control-control-NOT also called Toffoli gate :

\[ |c_1 \rangle \rightarrow |c_1 \rangle \]
\[ |c_2 \rangle \rightarrow |c_2 \rangle \]
\[ |c_2 \rangle \rightarrow |t \oplus c_1 c_2 \rangle \]

is equivalent to the following circuit made of CNOT, H, T and S:

\[ |c_1 \rangle \]
\[ |c_2 \rangle \]
\[ |t \rangle \rightarrow |t \oplus c_1 c_2 \rangle \]