Exercise 1  Dirac’s notation for vectors and matrices

(a) If \( |w\rangle \) is a vector and \( \alpha \) is a scalar, then
\[
(\alpha |w\rangle)^\dagger = \langle w| \alpha^* = \alpha^* \langle w|
\]
(you can check this in components). Moreover, we have linearity of transposition and complex conjugation :
\[
(\alpha |v\rangle + \beta |w\rangle)^\dagger = (\alpha |v\rangle)^\dagger + (\beta |w\rangle)^\dagger.
\]

(b) Then we get
\[
\langle v| = (|v\rangle)^\dagger = (v_1 |e_1\rangle + v_2 |e_2\rangle + \cdots + v_N |e_N\rangle)^\dagger
= v_1^* \langle e_1| + v_2^* \langle e_2| + \cdots + v_N^* \langle e_N|.
\]

(c) If \( \langle v| = \sum_{i=1}^N v_i^* \langle e_i| \) and \( |w\rangle = \sum_{j=1}^N w_j |e_j\rangle \), then
\[
\langle v|w\rangle = \sum_{i=1}^N \sum_{j=1}^N v_i^* w_j \langle e_i|e_j\rangle
= \sum_{i=1}^N \sum_{j=1}^N v_i^* w_j \delta_{ij}
= \sum_{i=1}^N v_i^* w_i.
\]

(d) For \( \vec{v} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \), we have \( \|\vec{v}\|^2 = \vec{v}^T \cdot \vec{v} \), so \( \|\vec{v}\|^2 = \alpha^* \alpha + \beta^* \beta \). On the other hand,
\( \langle v|v\rangle = \alpha^* \alpha + \beta^* \beta \) also by (c).

(e) Using components we have :
\[
|e_k\rangle \langle e_l| = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}
\begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}_{l-\text{th pos}} = \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}
\]
(f) Thus,

\[ A = \sum_{k,l} a_{kl} |e_k\rangle \langle e_l|.\]

So,

\[ \langle e_i| A |e_j\rangle = \sum_{l,t} a_{kl} \langle e_i|e_k\rangle \langle e_l|e_j\rangle = \sum_{l,t} a_{kl} \delta_{ik} \delta_{lj} = a_{ij}.\]

(g) From the point (e), we have

\[ I = \sum_{i=1}^{N} |e_i\rangle \langle e_i|.\]

Indeed, \( |e_i\rangle \langle e_i|\) is the matrix with 1 at the \( i\)-th row and \( i\)-th column and zeros elsewhere. This is called the closure relation.

(h) First note that the closure relation is valid for any orthonormal basis. Indeed, if \( \{ |\varphi_i\rangle \}_{i=1}^{N} \) are orthonormal, there exists a unitary basis change (a “rotation”) such that

\[ |\varphi_i\rangle = U |e_i\rangle, \quad \langle \varphi_i| = \langle e_i| U^\dagger.\]

Then from \( I = \sum_{i=1}^{N} |e_i\rangle \langle e_i| \) we get :

\[ UIU^\dagger = \sum_{i=1}^{N} U |e_i\rangle \langle e_i| U^\dagger
\]

\[ I = \sum_{i=1}^{N} |\varphi_i\rangle \langle \varphi_i|.\]

Now, from \( \alpha_i |\varphi_i\rangle = A |\varphi_i\rangle \) we get

\[ \sum_{i=1}^{N} \alpha_i |\varphi_i\rangle \langle \varphi_i| = \sum_{i=1}^{N} A |\varphi_i\rangle \langle \varphi_i| = A \sum_{i=1}^{N} |\varphi_i\rangle \langle \varphi_i| = AI = A.\]

Exercise 2 Tensor Product in Dirac’s notation

(a) By distributivity of the tensor product (first two properties), it follows that :

\[ |v\rangle_1 \otimes |w\rangle_2 = \left( \sum_{i=1}^{N} v_i |e_i\rangle_1 \right) \otimes \left( \sum_{j=1}^{M} w_j |f_j\rangle_2 \right) = \sum_{i=1}^{N} \sum_{j=1}^{M} v_i w_j |e_i\rangle_1 \otimes |f_j\rangle_2.\]
(b) Take two vectors $|e_i, f_j\rangle$ and $|e_k, f_l\rangle$ of $\mathcal{H}_1 \otimes \mathcal{H}_2$. Then by definition of the inner product:

$$\langle e_k, f_l|e_i, f_j\rangle = \langle e_k|e_i\rangle \langle f_l|f_j\rangle = \delta_{kl} \cdot \delta_{ij} = \delta_{(k,l);(i,j)}.$$ 

So this equals one if and only if $(k, l) = (i, j)$ and zero otherwise. This means that

$$\{ |e_i, f_j\rangle ; i = 1 \ldots N; j = 1 \ldots N \}$$ 

is an orthonormal basis of $\mathcal{H}_1 \otimes \mathcal{H}_2$.

The dimension equals the number of basis vectors, so is $NM$, the product of dim $\mathcal{H}_1$ and dim $\mathcal{H}_2$.

(c) We apply the definition

$$A \otimes B |\Psi\rangle = \sum_{i,j} \psi_{ij} A |e_i\rangle \otimes B |f_j\rangle$$

to $|\Psi\rangle = |e_k, f_l\rangle$. So $\psi_{ij} = 1$ for $(i, j) = (k, l)$ and 0 otherwise. This means:

$$A \otimes B |e_k, f_l\rangle = A |e_k\rangle \otimes B |f_l\rangle$$

and multiplying by $\langle e_i, f_j|$, we find:

$$\langle e_i, f_j| A \otimes B |e_k, f_l\rangle = \left( \langle e_i| \otimes \langle f_j| \right) \left( A |e_k\rangle \otimes B |f_l\rangle \right)$$

$$= \langle e_i| A |e_k\rangle \langle f_j| B |f_l\rangle$$

$$= a_{ik} b_{jl}.$$ 

(d) The formulas follow by translating the formulas found in (a) and (c) to the component notation.

**Exercise 3** *Billiard ball model of classical computation*

Discussed in exercise session. The billiard ball model of computation shows that it is possible to compute any Boolean function with elastic collisions between balls in billiards. This is a dissipationless computation which moreover conserves the number of balls (mass). Note that only collisions between pairs of balls are needed (or triple collision needed). This is in contrast to CCNOT and Fredkin which use 3 bits. But note that Fredkin conserves the “number of one’s”.

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