Problem 1. (a) Suppose $U$ and $V$ are binary random variables. The joint distribution induced on $(U, V)$ is given as

$$p_{UV}(u, v) = \begin{cases} 
\frac{1}{3}, & (u, v) = (0, 0) \\
\frac{1}{3}, & (u, v) = (1, 0) \\
\frac{1}{3}, & (u, v) = (1, 1) \\
0, & (u, v) = (0, 1)
\end{cases}$$

Find the Slepian-Wolf rate region for $(U, V)$ pair.

(b) Now suppose we have a binary additive MAC channel with inputs $X_1, X_2$ and output $Y$. The random variables $X_1$ and $X_2$ can take values in the set $\{0, 1\}$ and $Y$ can take values in the set $\{0, 1, 2\}$. The relationship between $X_1, X_2$ and $Y$ is given as

$$Y = X_1 + X_2.$$ 

Find the capacity region for this MAC.

(c) Now, the aim is to design a communication system that first compresses the source into a bitstream and then employs some channel coding technique to achieve reliable communication. The scheme is given as follows.

Here, SW-enc represents Slepian-Wolf encoder for the source $U,V$ of length $L$ which outputs a bitstream of length $LR_U, LR_V$ respectively. Later, the bitstreams $J_u$ and $J_v$ are encoded by channel encoders (Ch-enc) and then passed through the multiple access channel. As usual, from $Y^N$, bitstreams $\hat{J}_u$ and $\hat{J}_v$ are estimated by a channel decoder. Finally, the estimated bitstreams are decoded by Slepian-Wolf decoders to obtain $\hat{U}^L$ and $\hat{V}^L$.

For the sources described in part (a) and channel described in part (b), what is the maximum value that $L/N$ can take for a reliable communication?

(d) Consider now an uncoded scheme with the same sources and same channel where $X_1 = U$ and $X_2 = V$. Note that in this scheme, $L = N = 1$. Can $(U, V)$ be recovered from $Y$? Can the value $L/N$ of this scheme be achieved by schemes as in part (c)?

Problem 2. Consider the multiplicative multiple access channel $Y = X_1X_2$. Find the capacity region when
(a) $X_1 \in \{0, 1\}, X_2 \in \{1, 2\}$.
(b) $X_1 \in \{0, 1\}, X_2 \in \{1, 2, 3\}$.
(c) $X_1 \in \{1, 2\}, X_2 \in \{1, 2\}$. 