PROBLEM 1. Show that, if $H$ is the parity-check matrix of a code of length $n$, then the code has minimum distance $d$ if every $d - 1$ rows of $H$ are linearly independent and some $d$ rows are linearly dependent.

PROBLEM 2. In this problem we will show that there exists a binary linear code which satisfies the Gilbert–Varshamov bound. In order to do so, we will construct a $n \times r$ parity-check matrix $H$ and we will use Problem 1.

(a) We will choose rows of $H$ one-by-one. Suppose $i$ rows are already chosen. Give a combinatorial upper-bound on the number of distinct linear combinations of these $i$ rows taken $d - 2$ or fewer at a time.

(b) Provided this number is strictly less than $2^r - 1$, can we choose another row different from these linear combinations, and keep the property that any $d - 1$ rows of the new $(i + 1) \times r$ matrix are linearly independent?

(c) Conclude that there exists a binary linear code of length $n$, with at most $r$ parity-check equations and minimum distance at least $d$, provided

$$1 + \binom{n-1}{1} + \cdots + \binom{n-1}{d-2} < 2^r. \tag{1}$$

(d) Show that there exists a binary linear code with $M = 2^k$ distinct codewords of length $n$ provided $M \sum_{i=0}^{d-2} \binom{n-1}{i} < 2^n$.

PROBLEM 3. The weight of a binary sequence of length $N$ is the number of 1’s in the sequence. The Hamming distance between two binary sequences of length $N$ is the weight of their modulo 2 sum. Let $x_1$ be an arbitrary codeword in a linear binary code of block length $N$ and let $x_0$ be the all-zero codeword. Show that for each $n \leq N$, the number of codewords at distance $n$ from $x_1$ is the same as the number of codewords at distance $n$ from $x_0$.

PROBLEM 4. Let $W : \{0, 1\} \longrightarrow \mathcal{Y}$ be a channel where the input is binary and where the output alphabet is $\mathcal{Y}$. The Bhattacharyya parameter of the channel $W$ is defined as

$$Z(W) = \sum_{y \in \mathcal{Y}} \sqrt{W(y|0)W(y|1)}.$$ 

Let $X_1, X_2$ be two independent random variables uniformly distributed in $\{0, 1\}$ and let $Y_1$ and $Y_2$ be the output of the channel $W$ when the input is $X_1$ and $X_2$ respectively, i.e.,

$$P_{Y_1,Y_2|X_1,X_2}(y_1, y_2|x_1, x_2) = W(y_1|x_1)W(y_2|x_2).$$

Define the channels $W^- : \{0, 1\} \longrightarrow \mathcal{Y}^2$ and $W^+ : \{0, 1\} \longrightarrow \mathcal{Y}^2 \times \{0, 1\}$ as follows:

- $W^-(y_1, y_2 | u) = P[Y_1 = y_1, Y_2 = y_2 | X_1 \oplus X_2 = u_1]$ for every $u_1 \in \{0, 1\}$ and every $y_1, y_2 \in \mathcal{Y}$, where $\oplus$ is the XOR operation.
\[ W^+(y_1, y_2, u_1|u_2) = \mathbb{P}[Y_1 = y_1, Y_2 = y_2, X_1 \oplus X_2 = u_1|X_2 = u_2] \] for every \( u_1, u_2 \in \{0, 1\} \) and every \( y_1, y_2 \in \mathcal{Y} \).

(a) Show that \( W^- (y_1, y_2|u_1) = \frac{1}{2} \sum_{u_2 \in \{0, 1\}} W(y_1|u_1 \oplus u_2)W(y_2|u_2) \).

(b) Show that \( W^+ (y_1, y_2, u_1|u_2) = \frac{1}{2} W(y_1|u_1 \oplus u_2)W(y_2|u_2) \).

(c) Show that \( Z(W^+) = Z(W)^2 \).

For every \( y \in \mathcal{Y} \) define \( \alpha(y) = W(y|0) \), \( \beta(y) = W(y|1) \) and \( \gamma(y) = \sqrt{\alpha(y)\beta(y)} \).

(d) Show that
\[
Z(W^-) = \sum_{y_1, y_2 \in \mathcal{Y}} \frac{1}{2} \sqrt{\left( \alpha(y_1)\alpha(y_2) + \beta(y_1)\beta(y_2) \right) \left( \alpha(y_1)\beta(y_2) + \beta(y_1)\alpha(y_2) \right)}.
\]

(e) Show that for every \( x, y, z, t \geq 0 \) we have \( \sqrt{x + y + z + t} \leq \sqrt{x} + \sqrt{y} + \sqrt{z} + \sqrt{t} \). Deduce that
\[
Z(W^-) \leq \frac{1}{2} \left( \sum_{y_1, y_2 \in \mathcal{Y}} \alpha(y_1)\gamma(y_2) \right) + \frac{1}{2} \left( \sum_{y_1, y_2 \in \mathcal{Y}} \alpha(y_2)\gamma(y_1) \right)
+ \frac{1}{2} \left( \sum_{y_1, y_2 \in \mathcal{Y}} \beta(y_2)\gamma(y_1) \right) + \frac{1}{2} \left( \sum_{y_1, y_2 \in \mathcal{Y}} \beta(y_1)\gamma(y_2) \right) \quad (2)
\]

(f) Show that every sum in (2) is equal to \( Z(W) \). Deduce that \( Z(W^-) \leq 2Z(W) \).