Coupled Graphical Models and Their Thresholds

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Outline

- **Threshold saturation in convolutional LDPC codes.**
  - Feldstroem and Zigangirov 1999,
  - Engdahl, Lentmaier, and Zigangirov, 1999
  - Tanner, Sridhara, Sridhara, Fuja, and Costello, 2004
  - Sridhara, Lentmaier, Costello, and Zigangirov, 2004
  - [BEC]
  - Lentmaier, Sridharan, Zigangirov, and Costello, 2004
  - [general channels]
  - Kudekar, Richardson, Urbanke, 2010  [BEC]

- **Generality and underpinnings of this phenomenon in wide range of graphical models beyond coding theory.**
Spatially Coupled LDPC Ensemble.
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Extended EXIT Curve

(3, 6, L) ensemble; L increasing

BP Threshold
Spatially Coupled LDPC Ensemble.

Extended EXIT Curve

(3, 6, L) ensemble; L increasing
Spatially Coupled LDPC Ensemble.

Extended EXIT Curve

(3, 6, L) ensemble; L increasing
Spatially Coupled LDPC Ensemble.

Extended EXIT Curve

(3, 6, L) ensemble; L increasing
The fine structure:

Coupled density evolution equus:

\[ x_i = \epsilon \left( 1 - \frac{1}{w} \sum_{j=0}^{w-1} \left( 1 - \frac{1}{w} \sum_{k=0}^{w-1} x_{i+j-k} \right)^{x-1} \right)^{1-1} \]

Wiggles do not vanish by increasing L.

They do vanish by increasing width of coupling.
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Coupled density evolution equs:

\[ x_i = \epsilon \left( 1 - \frac{1}{w} \sum_{j=0}^{w-1} \left( 1 - \frac{1}{w} \sum_{k=0}^{w-1} x_{i+j-k} \right)^{r-1} \right)^{1-1} \]

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Decoder gets stuck at fixed points with kink profiles \( x_i \):
Threshold Saturation: is it a general phenomenon?

- Investigate it in other classes of graphical models.
  - Curie-Weiss model
  - Random Curie-Weiss model
  - Random k-SAT, xor-SAT
  - Random q-COL
  - Source coding
  - ...

- We show that the phenomenon is indeed very natural/general.
Curie-Weiss Model.

Take a **complete graph** and associate to vertex $i$ a variable $s_i \in \{-1, +1\}$.

$$H_N(s) = - \frac{J}{N} \sum_{(i,j)} s_is_j - h \sum_i s_i$$
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$$H_N(s) = -\frac{J}{N} \sum_{(i,j)} s_i s_j - h \sum_i s_i$$

$$\text{Prob}(s) = \frac{e^{-H_N(s)}}{Z_N} \quad \text{with} \quad Z_N = \sum_s e^{-H_N(s)}$$

Magnetization: $m = \frac{1}{N} \sum_{i=1}^N s_i$ (self-averaging)
Solving the Curie-Weiss model:

\[ Z_N \approx \int_{-1}^{+1} dm \, e^{-N\phi(m)} \text{ with } \phi(m) = -\frac{J}{2}m^2 - hm - H_2\left(\frac{1+m}{2}\right) \]
Solving the Curie-Weiss model:

\[ Z_N \approx \int_{-1}^{+1} dm \, e^{-N\phi(m)} \quad \text{with} \quad \phi(m) = -\frac{J}{2}m^2 - hm - \mathcal{H}_2\left(\frac{1+m}{2}\right) \]

Magnetization is given by equilibrium positions: \( \frac{\partial}{\partial m} \phi(m) = 0 \).
A fixed point equation:

This yields \( Jm - \frac{1}{2} \ln \frac{1+m}{1-m} + h = 0 \) or equivalently,

\[
m = \tanh(Jm + h)
\]

This is the CW equation, analog to DE fixed point equation.

The analog of the Extended EXIT curve is Van der Waals curve.
Coupled Curie-Weiss Model.
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At position $z$ we have complete graph with $N$ vertices and $s_{i,z}$ attached to them.
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$$J \approx \frac{J}{4N} \times N + \frac{J}{2N} \times (N - 1) + \frac{J}{4N} \times N$$
The coupled fixed point equations:

\[
\frac{J}{4} \frac{D^2}{Dz^2} m(z) = -Jm(z) + \frac{1}{2} \ln \frac{1+m(z)}{1-m(z)} - h
\]

with: \[\frac{D^2}{Dz^2} m(z) = m(z - 1) - 2m(z) + m(z + 1).\]

\[m(-L - 1) = m_-\]
\[m(L + 1) = m_+\]
The coupled fixed point equations:

\[ \frac{J}{4} \frac{D^2}{Dz^2} m(z) = -J m(z) + \frac{1}{2} \ln \frac{1 + m(z)}{1 - m(z)} - h \equiv \frac{\partial \phi(m(z))}{\partial m(z)} \]

with:

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\[ m(-L - 1) = m_- \]

\[ m(L + 1) = m_+ \]

This is a discrete \textbf{Newtonian equation of motion}. But with a “\textit{wrong sign}” and with boundary conditions.
Kink profiles and rolling balls in inverted potentials:
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\[ J = 1.1, \ h = 0 \]
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\[ J = 1.1, h = 0 \]

\[ J = 1.1, h = 0.55 \]
Generalized Van-der-Waals curve:

\[ m = \frac{1}{2L+1} \sum_{z=-L}^{L} m(z) \]

Exactly as in the LDPC case the wiggles can be associated to the kinks.

They vanish as the width of coupling window increases.
Coloring Random Graphs

$G(n, p)$: $n$ vertices, each of the $\binom{n}{2}$ possible edges occur with probability $p$

$q$-coloring: can we color vertices of $G$ with $q$ colors such that no adjacent vertices have the same color?

We are interested in Erdoes-Renyi graphs with $p = \frac{c}{n}$. 
Survey propagation formalism:

Coloring can be cast in the formalism of statistical physics.

This allows to derive a set of *fixed point equations* involving messages propagating on the edges of the graph.
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\[ q = 4 \]

- **Messages** = (random) probabilities of warning events.
- **SP** is a density evolution for these random probabilities.
Large $q$ analysis:

As $q \to +\infty$ we get a one dimensional equation for the average warning probability:

$$\phi = c e^{-\phi} \frac{(1-e^{-\phi})q-1}{1-(1-e^{-\phi})q}$$

Asymptotics of thresholds:

$$c_q \approx 2q \ln q$$

$$c_{SP} \approx q \ln q$$
**Coupled q-coloring Model.**

**Uncoupled model:** the $pn$ edges of Erdoes-Rényi graph are represented by degree 2 check nodes connected uniformly at random with the $n$ vertices.
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**Uncoupled model:** the $pn$ edges of Erdoes-Rényi graph are represented by degree 2 check nodes connected uniformly at random with the $n$ vertices.

**Coupled model:**
Large $q$ analysis of coupled model:

SP equations for the coupled system become a set of one-dimensional equations:

$$
\phi(z) = c F_q \left( \frac{1}{2w + 1} \sum_{k=-w}^{w} \phi(z + k) \right)
$$

Here $F_q(\phi) = e^{-\phi} \frac{(1 - e^{-\phi})q-1}{1 - (1 - e^{-\phi})q}$

$L = 20, w = 3$ and $q = 5$
“Van der Waals”/EXIT curve and threshold saturation:

\[ \bar{\phi} = \frac{1}{2L+1} \sum_z \phi(z) \]

For \( L >> w \) and \( q \to +\infty \):

\[ c_{SP} = c_q = 2q \ln q \]
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Conclusions and Open Directions

- **Algorithms**
  - Behavior of algorithms in uncoupled versus coupled
  - Improve lower bounds on *threshold of uncoupled*

- **Rigorous analysis**
  - MAP thresholds are asymptotically equal
  - Many open issues: *threshold saturation*, existence of kinks, ...