Homework 9

Exercise 1. Let us consider the two-dimensional Black-Scholes equation

\[
\begin{align*}
    dX_1 &= X_1 \left( \mu_1 \, dt + \sigma_{11} \, dB^{(1)}_t + \sigma_{12} \, dB^{(2)}_t \right), \quad X_1(0) = x_1(0), \\
    dX_2 &= X_2 \left( \mu_2 \, dt + \sigma_{21} \, dB^{(1)}_t + \sigma_{22} \, dB^{(2)}_t \right), \quad X_2(0) = x_2(0),
\end{align*}
\]

where \( x_1(0), x_2(0) > 0, B = (B^{(1)}, B^{(2)}) \) is a standard two-dimensional Brownian motion, \( \mu_1, \mu_2 \in \mathbb{R} \) and \( \sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22} > 0. \)

a) Give a necessary and sufficient condition on \( \sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22} \) so that the diffusion \( X = (X_1, X_2) \) is non-degenerate on the set \( D = \{ x \in \mathbb{R}^2 : x_1 \neq 0 \text{ and } x_2 \neq 0 \} \).

b) When this condition is satisfied, compute the martingale \( M \) which serves as a basis for the definition of the probability measure \( \tilde{P}_T \) under which the processes \( X_1, X_2 \) are martingales (up to time \( T \)).

Exercise 2. (Poisson’s equation)

Let \( D \) be an open and bounded domain in \( \mathbb{R}^n \) and \( \partial D \) be its (smooth) boundary. We assume that the following result is known: given \( g \in C(D) \), there exists a unique \( u \in C^2(D) \) satisfying

\[
\begin{align*}
    \frac{1}{2} \Delta u(x) &= -g(x), \quad x \in D, \\
    u(x) &= 0, \quad x \in \partial D.
\end{align*}
\]

Let now \( (B^x_t, t \in \mathbb{R}_+) \) be an \( n \)-dimensional Brownian motion starting at point \( x \in D \) at time \( t = 0 \) and let

\[ \tau = \inf\{ t > 0 : B^x_t \notin D \} \]

be the first exit time of \( B^x_t \) from \( D \) (notice that \( \tau \) depends on the starting point \( x \)).

a) Show that

\[ u(x) = \mathbb{E} \left( \int_0^\tau g(B^x_s) \, ds \right), \quad x \in D. \]

b) Let now \( D = B(0,1) \) be the unit ball in \( \mathbb{R}^n \) and let us define

\[ u(x) = \frac{1 - \|x\|^2}{n}. \]

Obviously, \( u(x) = 0 \) on \( \partial D = \{ x \in \mathbb{R}^n : \|x\| = 1 \} \). Find the function \( g \) such that \( u \) satisfies the above PDE.

c) Application: deduce the value of \( \mathbb{E}(\tau) \) as a function of \( x \).

d) In the special case where \( x = 0 \), how does this value behave as the dimension \( n \) increases?

e) Can you find an intuitive explanation for this last observation?