Exercise 1. Let $B$ be a standard (one-dimensional) Brownian motion and $M$ be the martingale defined as

$$M_t = \int_0^t s \, dB_s, \quad t \in \mathbb{R}_+.$$ 

a) Compute the quadratic variation of $M$.

For $s \in \mathbb{R}_+$, let us define

$$\tau(s) = \inf\{t \geq 0 : \langle M \rangle_t \geq s\}.$$

b) Compute explicitly $\tau(s)$.

Let $W$ be the process defined as

$$W_s = M(\tau(s)), \quad s \in \mathbb{R}_+.$$

c) Let $s_2 \geq s_1 \geq 0$. Compute $\mathbb{E}(W_{s_2} - W_{s_1})$ and $\mathbb{E}((W_{s_2} - W_{s_1})^2)$.

d) Do you have an idea of what type of process $W$ could be?

Exercise 2. Let $B$ be a standard (one-dimensional) Brownian motion and let $f : \mathbb{R} \to \mathbb{R}$ be the function defined as $f(x) = |x|$, $x \in \mathbb{R}$. Applying formally Ito-Doeblin’s formula to $f(B_t)$, neglecting the fact that $f$ is not twice continuously differentiable at $x = 0$ (not even once, actually), gives

$$|B_t| = \int_0^t \text{sgn}(B_s) \, dB_s + 0 \quad \text{a.s.}, \quad \forall t \in \mathbb{R}_+, \quad (1)$$

as

$$f'(x) = \text{sgn}(x) = \begin{cases} +1, & \text{if } x > 0, \\ -1, & \text{if } x < 0, \end{cases}$$

and $f''(x) = 0$, for all $x \neq 0$.

a) Question: can the above formula possibly hold? Justify your answer.

Let us now define

$$L_t = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{\{|B_s|<\varepsilon\}} \, ds.$$

$L_t$ can be thought of as the “time spent in $x = 0$ by the Brownian motion over the period $[0, t]$”.

b) Fact: $L_t$ is typically non-zero. Is this fact surprising to you? Justify your opinion.

It turns out that $L_t$ is the missing term in (1), that is, we actually have

$$|B_t| = \int_0^t \text{sgn}(B_s) \, dB_s + L_t \quad \text{a.s.}, \quad \forall t \in \mathbb{R}_+.$$ 

c) Show that

$$\mathbb{E}(|B_t|) = \mathbb{E} \left( \int_0^t \text{sgn}(B_s) \, dB_s + L_t \right), \quad \forall t \in \mathbb{R}_+.$$ 

For the computation of $\mathbb{E}(L_t)$, you are allowed to permute limits and integrals without asking too many questions!