Homework 1

Exercise 0*. Let \((M_t, t \in \mathbb{R}_+\) be a continuous square-integrable martingale with respect to a filtration \((\mathcal{F}_t, t \in \mathbb{R}_+)\).

a) Show that \(\text{Cov}(M_t, M_s)\) is only a function of \(t \wedge s := \min(t, s)\), and not of \(t\) and \(s\) separately.

b) Show that \(\text{Var}(M_t) \geq \text{Var}(M_s)\), if \(t > s \geq 0\).

c) Let \((N_t, t \in \mathbb{R}_+)\) be another martingale with respect to the same filtration \((\mathcal{F}_t, t \in \mathbb{R}_+)\), such that \(\mathbb{E}(N_0) = \mathbb{E}(M_0)\) and \(N_t \geq M_t\) a.s., for all \(t \in \mathbb{R}_+\). Show that \(N_t = M_t\) a.s., for all \(t \in \mathbb{R}_+\).

Exercise 1. Let \((B_t, t \in \mathbb{R}_+)\) be a standard Brownian motion and let

\[
M_t = \int_0^t s \, dB_s, \quad N_t = \int_0^t s \, dM_s.
\]

a) Compute the quadratic variation of \(M\).

b) Compute the quadratic variation of \(N\).

c) Knowing that the process \(N\) may be written as \(N_t = \int_0^t f(s) \, dB_s\), can you deduce an expression for the function \(f(s)\)?

Exercise 2. Let \((B_t, t \in \mathbb{R}_+)\) be a standard Brownian motion and let

\[
M_t = \int_0^t e^s \, dB_s, \quad N_t = \int_0^t e^{-s} \, dM_s.
\]

a) Compute the quadratic variation of \(M\).

b) Compute the quadratic variation of \(N\).

c) What can you deduce on the process \(N\)?

Exercise 3. Let \((B_t, t \in \mathbb{R}_+)\) be a standard Brownian motion and let

\[
M_t = \int_0^t e^s \, dB_s, \quad N_t = \int_0^t e^{-s} \, dB_s.
\]

a) Compute the quadratic covariation of \(B\) and \(M\).

b) Deduce from a) the value of \(\mathbb{E}(B_t M_t)\).

c) Compute the quadratic covariation of \(M\) and \(N\).

d) Deduce from c) the value of

\[
\mathbb{E}\left( \int_0^t s \, dM_s \int_0^t s \, dN_s \right).
\]