Solutions 7

1. a) Ito-Doeblin’s formula applied to \( v(X_t) \) gives

\[
v(X_t) - v(x_0) = \int_0^t v'(X_s) dX_s + \frac{1}{2} \int_0^t v''(X_s) d\langle X \rangle_s
\]

\[
= \int_0^t v'(X_s) f(X_s) ds + \int_0^t v'(X_s) g(X_s) dB_s + \frac{1}{2} \int_0^t v''(X_s) g(X_s)^2 ds
\]

\[
= \int_0^t A v(X_s) ds + \int_0^t v'(X_s) g(X_s) dB_s,
\]

therefore the result.

b) From a), we deduce that

\[
\lim_{t \downarrow 0} \frac{1}{t} \mathbb{E}(v(X_t) - v(x_0)) = \lim_{t \downarrow 0} \frac{1}{t} \mathbb{E}\left( \int_0^t A v(X_s) ds \right) = A v(x_0)
\]

by the fact that \( A v \) is a continuous function and the mean value theorem.

c) Setting \( v(x) = x \) in the previous equality, we obtain

\[
\lim_{t \downarrow 0} \frac{1}{t} \mathbb{E}(X_t - x_0) = 1 \cdot f(x_0) + 0 \cdot \frac{1}{2} g(x_0)^2 = f(x_0).
\]

4) Noticing that \( (X_t - x_0)^2 = (X_t^2 - x_0^2) - 2x_0 (X_t - x_0) \), and applying b) with \( v(x) = x^2 \) and \( v(x) = x \), we obtain

\[
\lim_{t \downarrow 0} \frac{1}{t} \mathbb{E}((X_t - x_0)^2) = 2x_0 \cdot f(x_0) + 2 \cdot \frac{1}{2} g(x_0)^2 - 2x_0 \cdot f(x_0) = g(x_0)^2.
\]

2. a) By the class, we know that there exists a probability measure \( \tilde{P}_T \) under which \((X_t, t \in [0,T])\) is a Brownian motion starting at point \( x_0 \) at time \( t = 0 \). We also know that the premium of the option at time \( t < T \) when \( X_t = x \) is given by

\[
c(t, x) = \tilde{E}_T(\max(X_T^t - x, 0)).
\]

where \( (X_s^t, s \in [t,T]) \) is a Brownian motion starting at point \( x \) at time \( t \). Therefore,

\[
c(t, x) = \int_{\mathbb{R}} \max(y - K, 0) p_{t-t}(x - y) dy = \int_{\mathbb{R}} \max(x - y - K, 0) p_{t-t}(y) dy
\]

\[
= \int_{-\infty}^{x-K} (x - y - K) p_{t-t}(y) dy
\]

where \( p_{t-t} \) if the pdf of the \( \mathcal{N}(0, T-t) \) distribution. After integration, this can be further written as

\[
c(t, x) = (x - K) N \left( \frac{x-K}{\sqrt{T-t}} \right) + \sqrt{\frac{T-t}{2\pi}} \exp \left( -\frac{(x-K)^2}{2(T-t)} \right).
\]
Remark: Notice also that the premium \( c(t, x) \) satisfies the PDE

\[
c'_t(t,x) + \frac{1}{2} c''_{xx}(t,x) = 0, \quad c(T, x) = \max(x - K, 0).
\]

b) By the class, the hedging strategy \( (\phi_t, \ t \in [0, T]) \) is given by \( \phi_t = c'_x(t, X_t) \). Rather than deriving the last formula obtained for \( c(t, x) \), let us move one step back to obtain

\[
c'_x(t,x) = \frac{\partial}{\partial x} \int_{-\infty}^{x-K} (x - y - K) p_{T-t}(y) \, dy = \int_{-\infty}^{x-K} p_{T-t}(y) \, dy = N\left(\frac{x-K}{\sqrt{T-t}}\right).
\]

Reminder:

\[
\frac{\partial}{\partial x} \int_{a(x)}^{\infty} f(x, y) \, dy = f(x, a(x)) \, a'(x) + \int_{a(x)}^{\infty} f_x'(x, y) \, dy,
\]

and notice that the first term is zero in the computation of \( c'_x(t, x) \).

3. By the class, the hedging strategy \( (\phi_t, \ t \in [0, T]) \) is given by \( \phi_t = c'_x(t, X_t) \), where \( c(t, x) \) is the premium at time \( t < T \) when \( X_t = x \). Remember also that in the Black-Scholes model with time-independent coefficients, \( c(t, x) \) is given by

\[
c(t, x) = \int_{y(t, x)}^{\infty} dy \, p_{T-t}(y) \left( x e^{\sigma y - \frac{\sigma^2(T-t)}{2}} - K \right),
\]

where

\[
y(t, x) = \frac{1}{\sigma} \left( \log \frac{K}{x} + \frac{\sigma^2(T-t)}{2} \right)
\]

is defined such that the expression in parentheses in the above formula is equal to zero when we set \( y = y(t, x) \). Using again the above reminder, we obtain

\[
c'_x(t, x) = -p_{T-t}(y(t, x)) \left( x e^{\sigma y(t,x) - \frac{\sigma^2(T-t)}{2}} - K \right) y'_x(t, x)
\]

\[
+ \int_{y(t, x)}^{\infty} dy \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{y^2}{2(T-t)}} e^{\sigma y - \frac{\sigma^2(T-t)}{2}}.
\]

By the remark just made about \( y(t, x) \), the first term on the right-hand side is zero, so

\[
c'_x(t, x) = \int_{y(t, x)}^{\infty} dy \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(y-x)^2}{2(T-t)}} = \ldots = N(d_1(t, x)),
\]

where

\[
d_1(t, x) = \frac{1}{\sigma \sqrt{T-t}} \left( \log \left( \frac{x}{K} \right) + \frac{\sigma^2(T-t)}{2} \right)
\]

Remark: From the formula

\[
c(t, x) = x \, N(d_1) - K \, N(d_2),
\]

one could be tempted to deduce that “of course”

\[
c'_x(t, x) = N(d_1).
\]

Notice nevertheless that both \( d_1 \) and \( d_2 \) depend on \( x \), i.e., there is some cancellation going on here!