1. a) \( \langle M \rangle_t = \int_0^t s^2 \, ds = t^3/3 \).

b) \( \tau(s) = \inf \{ t > 0 : t^3/3 \geq s \} = \inf \{ t > 0 : t \geq (3s)^{1/3} \} = (3s)^{1/3} \).

c) \( \mathbb{E}(W_{s_2} - W_{s_1}) = \mathbb{E}(M((3s_2)^{1/3}) - M((3s_1)^{1/3})) = 0 \), since \( M \) is a stochastic integral. Using now the isometry property, we have:

\[
\mathbb{E}((W_{s_2} - W_{s_1})^2) = \mathbb{E}((M((3s_2)^{1/3}) - M((3s_1)^{1/3}))^2) = \mathbb{E}\left( \left( \int_{(3s_1)^{1/3}}^{(3s_2)^{1/3}} s \, dB_s \right)^2 \right)
\]
\[
= \int_{(3s_1)^{1/3}}^{(3s_2)^{1/3}} s^2 \, ds = s_2 - s_1.
\]

d) Being a Wiener integral, \( M \) is a Gaussian process. As the time change \( \tau(s) \) is deterministic, \( W \) is also Gaussian. One can check further that \( W \) is actually a standard Brownian motion.

2. a) No. We have already seen that the process

\[ M_t = \int_0^t \text{sgn}(B_s) \, dB_s \]

on the right hand-side is a standard Brownian motion, since it is a continuous local martingale with quadratic variation \( \langle M \rangle_t = t \). The process on the left-hand side being non-negative, the equality cannot hold.

b) Remembering that \( B \) has either \( +\infty \) or \( -\infty \) slope, the fact that \( L_t \) is non-zero is surprising: a process that moves with infinite speed either up or down should not spend too much time in a given place (namely \( x = 0 \) here). Nevertheless, the Brownian motion is full of surprises...

c) Let us first compute

\[
\mathbb{E}(|B_t|) = \int_{\mathbb{R}} dx \frac{1}{\sqrt{2\pi t}} \exp \left( -\frac{x^2}{2t} \right) = \sqrt{\frac{2}{\pi t}} \int_0^\infty dr \, r \exp \left( -\frac{r^2}{2t} \right)
\]
\[
= \sqrt{\frac{2}{\pi t}} \left( -t \exp \left( -\frac{t^2}{2t} \right) \right) \bigg|_{r=0}^{r=\infty} = \sqrt{\frac{2t}{\pi}}.
\]

On the right-hand side, the first term has expectation zero. Let us then compute, permuting limit, integral and expectation:

\[
\mathbb{E}(L_t) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t ds \, \mathbb{E}(1_{|B_s| < \varepsilon}) \, ds = \int_0^t ds \, \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \mathbb{P}(\{|B_s| < \varepsilon\}) \, ds.
\]

Notice that

\[
\mathbb{P}(\{|B_s| < \varepsilon\}) = \int_{-\varepsilon}^{\varepsilon} \frac{1}{\sqrt{2\pi s}} \exp \left( -\frac{x^2}{2s} \right) \, dx,
\]
so by the mean value theorem,
\[
\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \mathbb{P}(\{|B_s| < \varepsilon\}) = \frac{1}{\sqrt{2\pi s}}, \quad \forall s > 0.
\]

Finally, we obtain that
\[
\mathbb{E}(L_t) = \int_0^t ds \frac{1}{\sqrt{2\pi s}} = \sqrt{\frac{2t}{\pi}},
\]
which coincides with \(\mathbb{E}(|B_t|)\) computed above.

NB: You may have noticed that \(\mathbb{E}(L_t) > t\) for \(t\) sufficiently small. This indicates that \(L_t\) is not exactly the time spent by \(B\) in \(x = 0\) on the interval \([0, t]\). Actually, \(L_t\) is only the *density* evaluated in \(x = 0\) of the occupation measure of the process \(B\). As such, it may exceed \(t\), exactly like the density \(p(x)\) of a continuous random variable may exceed 1 in \(x = 0\) or elsewhere.