Solutions 3

1. First observe that the transition probabilities do not depend on the particular shape of the (convex) polygon, but just on the number of edges. Consider next a polygon with \( j + 3 \) edges initially. After the transition, the smallest possible polygon will have 3 edges and the largest possible polygon will have \( j + 4 \) edges. Thus the resulting polygons have \( k + 3 \) edges with \( 0 \leq k \leq j + 1 \). Since the transition is uniformly random, \( p_{jk} = \frac{1}{j+2} \), for \( 0 \leq k \leq j + 1 \).

2. Thus,

\[
E(X_n | X_{n-1} = j) = \frac{j+1}{j+2} \sum_{k=0}^{j+1} k p_{jk} = \frac{j+1}{j+2} \sum_{k=0}^{j+1} k = \frac{(j+1)(j+2)}{2(j+2)} = \frac{j+1}{2}
\]

and

\[
E(X_n) = \sum_{j \geq 0} E(X_n | X_{n-1} = j) \mathbb{P}(X_{n-1} = j) = \frac{1 + E(X_{n-1})}{2}
\]

Repeating this, we obtain \( E(X_n) = 1 - (1/2)^n + (1/2)^n E(X_0) \).

3. Consider \((X_n, n \geq 0)\) initialized with some initial distribution for \(X_0\). Repeating the above computation, we obtain

\[
E(s^{X_n} | X_{n-1} = j) = \frac{1}{j+2} \sum_{k=0}^{j+1} s^k = \frac{1}{j+2} \frac{1 - s^{j+2}}{1 - s}
\]

This implies that \( G_n(s) = \frac{1}{1 - s} \mathbb{E} \left( \frac{1 - s^{X_{n-1}+2}}{X_{n-1}+2} \right) \).

4. Now consider the process \((X_n, n \geq 0)\) initialized with \(X_0 \sim \pi\), where \(\pi\) is the stationary distribution. Since \(\pi = \pi P\) by definition, we have \(X_n \sim \pi\) and \(X_{n-1} \sim \pi\), so \(G(s) = E(s^{X_n}) = E(s^{X_{n-1}})\) for all \(n \geq 1\), and by part 3, we also have

\[
G(s) = \frac{1}{1 - s} \mathbb{E} \left( \frac{1 - s^{X_{n-1}+2}}{X_{n-1}+2} \right)
\]

where \(\mathbb{E}\) is taken with respect to \(\pi\).

Differentiating with respect to \(s\), we obtain

\[
G'(s) = \frac{1}{(1 - s)^2} \mathbb{E} \left( \frac{1 - s^{X_{n-1}+2}}{X_{n-1}+2} \right) - \frac{1}{1 - s} \mathbb{E} \left( \frac{X_{n-1} + 2}{X_{n-1}+2} s^{X_{n-1}+1} \right)
\]

\[
= \frac{1}{1 - s} G(s) - \frac{1}{1 - s} s G(s) = G(s)
\]

One checks also that \(G(1) = 1\) (using Bernoulli-L’Hospital’s rule), so the solution of this differential equation is \(G(s) = e^{s-1}\).

5. \(G_Y(s) = \sum_{k \geq 0} \lambda^k e^{-\lambda} s^k / k! = e^{-\lambda} \sum_{k \geq 0} (s\lambda)^k / k! = e^{\lambda(s-1)}\). Hence the stationary distribution of the Markov chain is Poisson with parameter 1.
2. a) Clearly, all states \( i \) s.t. \( i \) is not a power of 2 are transient. Indeed, if \( i \) is not a power of 2, then \( \mathbb{P}(X_n = i \text{ for some } n > 0 \mid X_0 = i) = 0 \).

Let us consider now the state 1. Then,

\[
f_{11}(n) = c^{n-1}(1 - c), \quad \text{for } n \geq 1.
\]

Hence,

\[
f_{11} = \sum_{n \geq 1} n f_{11}(n) = (1 - c) \sum_{n \geq 1} nc^{n-1} = \frac{1}{1 - c} < +\infty,
\]

which implies that the state 1 is positive-recurrent. As concerns the states \( \{2^k\}_{k \geq 1} \), they form together with state 1 an equivalence class of the Markov chain. Therefore, they are also positive-recurrent.

b) Let \( \pi \) be the stationary distribution (in case it exists). Then, by solving \( \pi = \pi P \), we obtain

\[
\begin{align*}
\pi_1 &= (1 - c) \sum_{i \in \mathbb{N}} \pi_{2^i} = 1 - c \\
\pi_{2^k} &= c \cdot \pi_{2^{k-1}} \quad k \geq 1 \\
\pi_i &= 0 \quad \text{otherwise}
\end{align*}
\]

Hence, the stationary distribution exists, is unique and is given by

\[
\begin{align*}
\pi_{2^k} &= (1 - c) \cdot c^k \quad k \geq 0 \\
\pi_i &= 0 \quad \text{otherwise}
\end{align*}
\]

c)-d) In general, by solving \( \pi = \pi P \), we obtain

\[
\begin{align*}
\pi_1 &= \sum_{k \in \mathbb{N}} (1 - p_{2^k}) \pi_{2^k} \\
\pi_{2^k} &= p_{2^{k-1}} \cdot \pi_{2^{k-1}} \quad k \geq 1 \\
\pi_i &= 0 \quad \text{otherwise}
\end{align*}
\]

Therefore, \( \pi_{2^k} = \prod_{j=0}^{k-1} p_{2^j} \pi_1 \), so the stationary distribution exists and is unique if and only if

\[
\sum_{k \in \mathbb{N}} \prod_{j=0}^{k-1} p_{2^j} < +\infty. \quad (1)
\]

(otherwise it would imply that \( \pi_1 = 0 \)).

Consider now the case \( c_k = p_{2^k} = 1 - \frac{1}{2^k + 1} \). We will show that \( \lim_{k \to \infty} \prod_{j=0}^{k} c_j \neq 0 \), which implies, through condition (1), that the stationary distribution does not exist. Note first that

\[
\lim_{k \to \infty} \prod_{j=0}^{k} c_j = 0 \iff \lim_{k \to \infty} \sum_{j=0}^{k} \log \frac{1}{c_j} = +\infty.
\]

In addition,

\[
\lim_{k \to \infty} \sum_{j=0}^{k} \log \frac{1}{c_j} = \lim_{k \to \infty} \sum_{j=0}^{k} \log (1 + 2^{-j}) \leq \lim_{k \to \infty} \sum_{j=0}^{k} 2^{-j} = 2 < +\infty,
\]

where we used the fact that \( \log(1 + x) \leq x \) for any \( x \in [0, 1] \). As a result, \( \sum_{k \in \mathbb{N}} \prod_{j=0}^{k-1} c_j = +\infty \), so the stationary distribution does not exist in this case.