1. For this exercise, let \((U_n, n \geq 1)\) be a sequence of i.i.d. \(\sim U([0, 1])\) random variables.

**First case.** \(X_0 = 0, Y_0 = 1\).

**a)** One coupling that maximizes the chances of \(X\) and \(Y\) to meet after the first step is described as follows:
\[
\begin{cases}
  \text{if } 0 \leq U_{n+1} \leq \frac{1}{4} & \text{then } X_{n+1} = X_n + 1 \text{ and } Y_{n+1} = Y_n \\
  \text{if } \frac{1}{4} < U_{n+1} \leq \frac{1}{2} & \text{then } X_{n+1} = X_n \text{ and } Y_{n+1} = Y_n - 1 \\
  \text{if } \frac{1}{2} < U_{n+1} \leq \frac{3}{4} & \text{then } X_{n+1} = X_n \text{ and } Y_{n+1} = Y_n \\
  \text{if } \frac{3}{4} < U_{n+1} \leq 1 & \text{then } X_{n+1} = X_n - 1 \text{ and } Y_{n+1} = Y_n + 1
\end{cases}
\]

With this coupling, the probability that \(X\) and \(Y\) meet after one step is \(\frac{1}{2}\), which can be seen to be the maximum.

**b)** Let \(\xi_{n+1}\) be the random variable defined as
\[
\xi_{n+1} = \begin{cases}
  +1 & \text{if } 0 \leq U_{n+1} \leq \frac{1}{4} \\
  0 & \text{if } \frac{1}{4} < U_{n+1} \leq \frac{3}{4} \\
  -1 & \text{if } \frac{3}{4} < U_{n+1} \leq 1
\end{cases}
\]

If both \(X_{n+1} = X_n + \xi_{n+1}\) and \(Y_{n+1} = Y_n + \xi_{n+1}\), then the two chains never meet.

But another option is also to have \(X_{n+1} = X_n + \xi_{n+1}\) and \(Y_{n+1} = Y_n - \xi_{n+1}\).

**Variant: \(X_0 = 0, Y_0 = 2\).**

**a)** In this case, one coupling that maximizes the chances of \(X\) and \(Y\) to meet after the first step is:
\[
\begin{cases}
  \text{if } 0 \leq U_{n+1} \leq \frac{1}{4} & \text{then } X_{n+1} = X_n + 1 \text{ and } Y_{n+1} = Y_n - 1 \\
  \text{if } \frac{1}{4} < U_{n+1} \leq \frac{3}{4} & \text{then } X_{n+1} = X_n \text{ and } Y_{n+1} = Y_n \\
  \text{if } \frac{3}{4} < U_{n+1} \leq 1 & \text{then } X_{n+1} = X_n - 1 \text{ and } Y_{n+1} = Y_n + 1
\end{cases}
\]

With this coupling, the probability that \(X\) and \(Y\) meet after one step is \(\frac{1}{4}\), which can be seen to be the maximum \((NB: \text{This coupling can also be described with the random variable } \xi_{n+1} \text{ above: } X_{n+1} = X_n + \xi_{n+1} \text{ and } Y_{n+1} = Y_n - \xi_{n+1})\).

**b)** In this case, only the coupling \(X_{n+1} = X_n + \xi_{n+1}\) and \(Y_{n+1} = Y_n + \xi_{n+1}\) ensures that the walks never meet. There is no other coupling guaranteeing this property.