**Homework 9 (due Wednesday, May 11)**

**Exercise 1. a)** Preliminary question. Consider the random walk on $\mathbb{Z}$ with transition probabilities $\psi_{i,i \pm 1} = 1/2$. Does this chain admit a stationary distribution? Is it positive-recurrent?

In this problem, we use the Metropolis-Hastings rule to bias the simple random walk so that the stationary distribution of the new walk $(X_n, n \geq 0)$ equals

$$
\pi_i = \frac{e^{-ai^2}}{\sum_{i=-\infty}^{+\infty} e^{-ai^2}}
$$

where $a > 0$ is a parameter. Moves $i \rightarrow j = i \pm 1$ are proposed with probability $\psi_{ij}$ and accepted with probability $a_{ij} = \min(1, \pi_j \psi_{ji} / \pi_i \psi_{ij})$.

b) Give the transition probabilities $p_{ij} = P(X_1 = j \mid X_0 = i)$ of the final chain for all $i$ and $j$.

c) Show that for this chain:

$$
\lim_{n \to +\infty} P(X_n = j \mid X_0 = i) = \pi_j, \quad \forall i \in \mathbb{Z}.
$$

For the next question, we consider two coupled walks $(X_n, n \geq 0)$ and $(Y_n, n \geq 0)$ on $\mathbb{Z}$. The walks are coupled in the following way:

- At each time step $n \geq 0$, we draw a common uniform random variable $\xi_n \in \{+1, -1\}$ and for each walk we propose the moves $X_n \rightarrow X_n + \xi_n$ and $Y_n \rightarrow Y_n + \xi_n$.
- Each move is accepted or rejected according to the Metropolis-Hastings rule of question b).

We define the coalescence time (a random variable)

$$
T = \inf \{n : X_n = Y_n \text{ given that } X_0 = z, Y_0 = z + d\}
$$

where $z$ and $d$ are strictly positive integers. We consider only the case $z > d$.

d) What is the smallest possible coalescence time? Compute the probability that the coalescence time takes this smallest possible value.