Exercise 1. A complete graph $K_m$ with $m$ vertices is a graph with one edge between all pairs of vertices. We consider the homogeneous random walk $(X_n, n \geq 0)$ on $K_m$ defined by the transition probability $P(X_{n+1} = j | X_n = i) = 1/(m - 1)$ for all vertices $j \neq i$. We also denote by $\mu_i^{(n)} = P(X_n = i), i \in K_m$ the probability distribution of a random walk at time $n$ with initial condition $X_0 = i_0$ and by $\nu_j^{(n)} = P(Y_n = j), j \in K_m$ the probability distribution of another independent random walk with initial condition $Y_0 = j_0$. The two initial vertices are fixed once for all and distinct, $i_0 \neq j_0$.

We now define the following homogeneous Markov process $((X_n, Y_n), n \geq 0)$ with state space $K_m \times K_m$ and transition probabilities

$$P(X_{n+1} = i', Y_{n+1} = j' | X_n = i, Y_n = j) = \begin{cases} \frac{1}{(m-1)^2} & \text{if } i \neq j \text{ and } i' \neq i, j' \neq j, \\ \frac{1}{m-1} & \text{if } j = i \text{ and } j' = i' \neq i, \\ 0 & \text{in all other cases.} \end{cases}$$

It is understood that this Markov process is conditioned on the initial condition $(X_0, Y_0) = (i_0, j_0)$.

a) Show that $(X_n, Y_n)$ is a coupling of the two probability distributions $\mu^{(n)}$ and $\nu^{(n)}$.

Hint: Recall the definition of coupling; you have to compute the marginals of $P(X_n = i, Y_n = j)$.

b) Consider the coalescence time $T = \inf \{ n \geq 1 \mid X_n = Y_n \}$ (a random variable). First show that for all $n \geq 1$:

$$P(T > n) = \left(1 - \frac{1}{m-1}\right)^n$$

and deduce from there that for all $n \geq 1$:

$$P(T = n) = \frac{1}{m-1} \left(1 - \frac{1}{m-1}\right)^{n-1}$$

Hint: rewrite the events $\{ T > n \}$ and $\{ T = n \}$ in terms of the $X$’s and $Y$’s.

c) Consider the total variation distance $\| \mu^{(n)} - \nu^{(n)} \|_{TV}$. Use the formula in point 1 of exercise 2 to show that

$$\| \mu^{(n)} - \nu^{(n)} \|_{TV} \leq e^{-n/(m-1)}$$

Hint: $1 - x \leq e^{-x}$ for all $x \geq 0$.

d) We remark that $\pi_i = 1/m, i \in K_m$ is a stationary distribution for the lazy random walk on the complete graph. Justify this remark. Use this remark and the previous bound to deduce

$$\| \mu^{(n)} - \pi \|_{TV} \leq e^{-n/(m-1)}$$

e) What happens in the (very) particular case $m = 2$?
Exercise 2. a) Let $\mu$ and $\nu$ be two distributions on a state space $S$ (i.e., $\mu_i, \nu_i \geq 0$ for every $i \in S$ and $\sum_{i \in S} \mu_i = \sum_{i \in S} \nu_i = 1$). Show that the following three definitions of the total variation distance between $\mu$ and $\nu$ are equivalent:

1. $\|\mu - \nu\|_{TV} = \frac{1}{2} \sum_{i \in S} |\mu_i - \nu_i|$.

2. $\|\mu - \nu\|_{TV} = \max_{A \subseteq S} |\mu(A) - \nu(A)|$, where $\mu(A) = \sum_{i \in A} \mu_i$ and $\nu(A) = \sum_{i \in A} \nu_i$.

3. $\|\mu - \nu\|_{TV} = \frac{1}{2} \max_{\phi : S \to [-1, +1]} |\mu(\phi) - \nu(\phi)|$, where $\mu(\phi) = \sum_{i \in S} \mu_i \phi_i$ and $\nu(\phi) = \sum_{i \in S} \nu_i \phi_i$.

*Hint:* The easiest way is to show that $1 \leq 2 \leq 3 \leq 1$.

b) Show that $\|\mu - \nu\|_{TV}$ is indeed a distance (i.e., that it is non-negative, that it is zero if and only if $\mu = \nu$, that it is symmetric and that the triangle inequality is satisfied).