Exercise 1. Let $X, Y$ be two Markov chains with following same transition probabilities:

$$p_{ij} = \begin{cases} 
1/2 & \text{if } j = i \\
1/4 & \text{if } j = i \pm 1 \\
0 & \text{otherwise}
\end{cases}$$

That is, $X$ and $Y$ are two versions of the symmetric lazy random walk on $\mathbb{Z}$. Let us assume now that $X_0 = 0$ and $Y_0 = 1$.

a) Using a random mapping representation, describe a coupling of $X$ and $Y$ such that these two chains meet with the highest probability after one step only. What is the value of this probability?

b) Using a random mapping representation, describe two different couplings of $X$ and $Y$ such that these two chains never meet.

Variant: Consider the same questions, but now with the initial conditions $X_0 = 0$ and $Y_0 = 2$. 