Exercise 1. (De Moivre-Laplace theorem)

Let \((S_n, n \in \mathbb{N})\) be the simple asymmetric random walk on \(\mathbb{Z}\), defined as
\[
S_0 = 0, \quad S_n = \xi_1 + \ldots + \xi_n, \quad n \geq 1,
\]
where the random variables \((\xi_n, n \geq 1)\) are i.i.d. with \(P(\xi_n = 1) = p \in ]0,1[\) and \(P(\xi_n = -1) = q = 1 - p\).

a) Compute \(E(S_n)\) and \(Var(S_n)\).

b) Using Stirling’s formula (valid for large values of \(n\)):
\[
n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n,
\]
show that for \(k\) in the neighborhood of \(n(2p - 1)\),
\[
p_{0,2k}^{(2n)} = P(S_{2n} = 2k | S_0 = 0) = \left(\frac{2n}{n+k}\right) p^{n+k} q^{n-k} \sim \frac{1}{\sqrt{4\pi npq}} \exp\left(-\frac{(k - n(2p - 1))^2}{4npq}\right),
\]
which is saying that for large \(n\), the distribution of \(S_n\) resembles that of a Gaussian random variable. This result is an early version of the celebrated central limit theorem.

c) Deduce from this (or rederive from scratch) that
\[
p_{0,0}^{(2n)} = P(S_{2n} = 0 | S_0 = 0) = \left(\frac{2n}{n}\right) p^n q^n \sim \frac{(4pq)^n}{\sqrt{\pi n}}.
\]

NB: The notation \(a_n \sim b_n\) means precisely
\[
\lim_{n \to \infty} \frac{a_n}{b_n} = 1.
\]

Exercise 2. Let \((\mathbf{S}_n, n \in \mathbb{N})\) be the simple symmetric random walk in two dimensions, that is,
\[
\mathbf{S}_0 = (0,0), \quad \mathbf{S}_n = \mathbf{\xi}_1 + \ldots + \mathbf{\xi}_n, \quad n \geq 1,
\]
where \((\mathbf{\xi}_n, n \geq 1)\) are i.i.d random variables such that
\[
P(\mathbf{\xi}_n = (+1,0)) = P(\mathbf{\xi}_n = (-1,0)) = P(\mathbf{\xi}_n = (0,+1)) = P(\mathbf{\xi}_n = (0,-1)) = \frac{1}{4}.
\]
Let us write \(\mathbf{S}_n = (X_n, Y_n)\).

a) What type of (unidimensional) random walks are \((X_n, n \in \mathbb{N})\) and \((Y_n, n \in \mathbb{N})\)?

b) Are these two random walks independent?

Define now \(U_n = X_n + Y_n\) and \(V_n = X_n - Y_n, n \in \mathbb{N}\). Again the same questions:

c) What type of (unidimensional) random walks are \((U_n, n \in \mathbb{N})\) and \((V_n, n \in \mathbb{N})\)?

d) Are these two random walks independent?

e) Deduce from this the value of \(P(\mathbf{S}_{2n} = (0,0) | \mathbf{S}_0 = (0,0))\). How does it behave for large \(n\)?