1. We have
\[
P \left( \left\{ \frac{S_n}{n} - \mu \geq \varepsilon \right\} \right) = P \left( \{|S_n - \mathbb{E}(S_n)| \geq n\varepsilon\} \right) \leq \frac{\text{Var}(S_n)}{n^2 \varepsilon^2}
\]
\[
= \frac{1}{n^2 \varepsilon^2} \sum_{i,j=1}^{n} \text{Cov}(X_i, X_j) \leq \frac{\sigma^2}{n^2 \varepsilon^2} \sum_{i,j=1}^{n} \exp(-|i - j|)
\]
\[
\leq \frac{2\sigma^2}{n \varepsilon^2} \sum_{k \in \mathbb{Z}} \exp(-|k|) \xrightarrow{n \to \infty} 0, \quad \text{as} \quad \sum_{k \in \mathbb{Z}} \exp(-|k|) < \infty.
\]

2. Using Chebychev’s inequality with \(\varphi(x) = x^4\), we obtain, for \(\varepsilon > 0\) fixed:
\[
P \left( \left\{ \left| \frac{S_n}{n} \right| \geq \varepsilon \right\} \right) = P \left( \{|S_n| \geq n\varepsilon\} \right) \leq \frac{\mathbb{E}(S_n^4)}{n^4 \varepsilon^4}
\]
Now, observe that \(\mathbb{E}(S_n^4) = \mathbb{E}((\sum_{j=1}^{n} X_j)^4) = \sum_{j,k,l,m=1}^{n} \mathbb{E}(X_j X_k X_l X_m)\), and the next key observation is that many terms are equal to zero in this sum, because of the fact that the \(X_j\) are independent and the fact that \(\mathbb{E}(X_j) = 0\). It turns out that only the terms where a) \(j = k = l = m\) or b) \(j = k\) and \(l = m\) or any pairing of that sort are non-zero. We therefore get
\[
\mathbb{E}(S_n^4) = \sum_{j=1}^{n} \mathbb{E}(X_j^4) + 3 \sum_{j \neq l} \mathbb{E}(X_j^2) \mathbb{E}(X_l^2) = nC + 3n(n-1)1 = O(n^2)
\]
This implies that for \(\varepsilon > 0\) fixed:
\[
P \left( \left\{ \left| \frac{S_n}{n} \right| \geq \varepsilon \right\} \right) \leq \frac{O(n^2)}{n^4 \varepsilon^4} = O \left( \frac{1}{n^2} \right).
\]
By the (first) Borel-Cantelli lemma, this implies that
\[
P \left( \left\{ \left| \frac{S_n}{n} \right| \geq \varepsilon \quad \text{infinitely often} \right\} \right) = 0
\]
i.e. that \(\frac{S_n}{n}\) converges almost surely to 0 as \(n \to \infty\).
3. We have \( E(X_n) = 0 \) and \( \text{Var}(X_n) = \frac{n}{\log(n)} \). So

\[
E(S_n) = \sum_{j=1}^{n} E(X_j) = 0 \quad \text{and} \quad \text{Var}(S_n) = \sum_{j=1}^{n} \text{Var}(X_j) = \sum_{j=2}^{n} \frac{j}{\log(j)}.
\]

Notice then that

\[
\sum_{j=2}^{n} \frac{j}{\log(j)} = \sum_{j=2}^{\sqrt{n}} \frac{j}{\log(j)} + \sum_{j=\lceil \sqrt{n} \rceil + 1}^{n} \frac{j}{\log(j)} \leq \sum_{j=2}^{\sqrt{n}} \frac{1}{\log(\sqrt{n})} \sum_{j=\lceil \sqrt{n} \rceil + 1}^{n} j \leq O(n) + O(n^2) \frac{1}{\log n},
\]

which implies that \( \frac{1}{n^2} \text{Var}(S_n) \to 0 \) as \( n \to \infty \). Therefore, by Chebychev’s inequality,

\[
P\left( \left\{ \frac{|S_n|}{n} > \epsilon \right\} \right) \leq \frac{\text{Var}(S_n)}{n^2 \epsilon^2} \to 0, \quad \forall \epsilon > 0,
\]

that is, \( \frac{S_n}{n} \to 0 \) as \( n \to \infty \).

To show that \( \frac{S_n}{n} \) does not converge to 0 a.s., we use the second Borel-Cantelli lemma. Let \( A_n = \{ \omega \in \Omega : |X_n(\omega)| \geq n \} \). We have

\[
\sum_{n \geq 1} P(A_n) = \sum_{n \geq 2} \frac{1}{n \log(n)} \approx \lim_{n \to \infty} \log(\log(n)) = \infty,
\]

where the approximation can be obtained using the integral test. Hence,

\[
P(A_n \text{ takes place infinitely often}) = 1,
\]

i.e.

\[
\frac{|X_n|}{n} \geq 1 \quad \text{takes place infinitely often, with probability 1.}
\]

Assume now that \( \frac{S_n}{n} \to 0 \) a.s. This would imply that for every \( \delta > 0 \),

\[
\frac{|S_n|}{n} < \delta \quad \text{takes place for all but finitely many values of } n, \text{ with probability 1.}
\]

But this leads to a contradiction, as

\[
\left| \frac{X_n}{n} \right| = \left| \frac{S_n - S_{n-1}}{n} \right| \leq \left| \frac{S_n}{n} \right| + \left| \frac{S_{n-1}}{n} \right| \leq \frac{|S_n|}{n} + \frac{|S_{n-1}|}{n-1}
\]

so taking \( \delta < 1/2 \), it cannot happen at the same time that \( |X_n/n| \geq 1 \) for an infinite number of values of \( n \) and that \( |S_n/n| < \delta \) for all but finitely many values of \( n \).