1. $X$ is a martingale, whatever the value of $p$:

$$
E(X_{n+1}|F_n) = E\left( X_n (q/p)^{\xi_{n+1}} | F_n \right) = X_n E\left( (q/p)^{\xi_{n+1}} \right) = X_n ((q/p)p + (p/q)q) = X_n.
$$

b) The third version of the martingale convergence theorem applies here, as the stopped martingale $X_{T\wedge n}$ is bounded. Let $p_N = \mathbb{P}\{\{S_T = N\}\}$ (notice that $\mathbb{P}\{\{S_T = 0\}\} = 1 - p_N$). Using the fact that $E(X_T) = E(X_0) = q/p$ (remember that $S_0 = 1$), we obtain

$$
q/p = E(X_T) = p_N (q/p)^N + (1 - p_N) 1,
$$

so $p_N = \frac{1 - (q/p)}{1 - (q/p)^N}$.

NB: This works only for $p \neq q$. In the case $p = q = 1/2$, the process $S$ itself is a martingale, and applying the optional stopping theorem gives $1 = E(S_0) = E(S_T) = N p_N + 0$, so $p_N = 1/N$.

2. a) Yes: $x \mapsto x^4$ is convex, so we conclude by Jensen’s inequality and the fact that $(S_n, n \geq 0)$ is a martingale.

b) Yes: $E(S_{n+1}^4 | F_n) = S_n^4 + 6S_n^2 + 1$, so $E(S_{n+1}^4 - (n + 1) | F_n) = (S_n^4 - n) + 6S_n^2 \geq S_n^4 - n$.

c) From b), $E(S_{n+1}^4) = E(S_n^4) + 6n + 1$, so by induction, $E(S_n^4) = 3n^2 - 2n$.

d) $\lim_{n \to \infty} \frac{E(S_n^2)}{n^2} = 3$. This could also be deduced directly from the central limit theorem, which states that $S_n/\sqrt{n}$ converges in distribution to an $\mathcal{N}(0,1)$ random variable, whose fourth moment is equal to 3.

3. a) The process is a martingale, as i) $E(|Y_n|) = E(Y_n) \leq (3/2)^n$ for all $n$; ii) $Y_n$ is $F_n$-measurable for all $n$, by definition, and

$$
E(Y_{n+1}|F_n) = \frac{1}{2} \left( \frac{3}{2} Y_n + \frac{1}{2} Y_n \right) = Y_n.
$$

b) $E(Y_n) = E(Y_0) = 1$, since $Y$ is a martingale and

$$
E(Y_{n+1}^2) = E(E(Y_{n+1}^2 | F_n)) = E \left( \frac{1}{2} \left( \frac{9}{4} Y_n^2 + \frac{1}{4} Y_n^2 \right) \right) = \frac{5}{4} E(Y_n^2),
$$

so by induction, $E(Y_n^2) = (5/4)^n$ (as $E(Y_0^2) = 1$) and $\text{Var}(Y_n) = (5/4)^n - 1$.

c) The process $Y$ is not confined to a bounded interval; it can take values between 0 and $\infty$.

d) As $\sup_{n \in \mathbb{N}} E(|Y_n|) = \sup_{n \in \mathbb{N}} E(Y_n) = 1 < \infty$, the second version of the martingale convergence theorem tells us that there exists a random variable $Y_\infty$ such that $Y_n \to Y_\infty$.

e) Notice that

$$
Z_{n+1} = \begin{cases} 
Z_n + \log(3/2), & \text{w.p. } 1/2, \\
Z_n + \log(1/2), & \text{w.p. } 1/2.
\end{cases}
$$
so \( \mathbb{E}(Z_n) = n \left( \log(3/2) + \log(1/2) \right)/2 = \frac{n}{2} \log(3/4) \). As \( \log(3/4) < 0 \), \( Z \) is a random walk with a negative drift. It can be shown that for any \( K > 0 \), there exists \( c > 0 \) such that

\[
P(Z_n \geq -K) \leq \exp(-cn), \quad \forall n,
\]

so \( Z_n \xrightarrow{n \to \infty} -\infty \) a.s., implying that \( Y_n = \exp(Z_n) \xrightarrow{n \to \infty} 0 = Y_\infty \) a.s.

f) The answer is no, as for every \( n \), \( \mathbb{E}(Y_\infty|\mathcal{F}_n) = \mathbb{E}(0|\mathcal{F}_n) = 0 \neq Y_n \).