Solutions 10

1. a) $X$ is not a martingale ($E(\xi_{n+1}|\mathcal{F}_n) = E(\xi_{n+1}) = 0 \neq \xi_n$).
   b) $X$ is not a martingale (except in the case where $a = 1$; notice however that it is neither a submartingale when $a > 1$, nor is it a supermartingale when $a < 1$, as $X$ can take positive and negative values).
   c) $X$ is not a martingale ($E(X_{n+1}|\mathcal{F}_n) = E(\xi_n|\mathcal{F}_n) + E(\xi_{n+1}|\mathcal{F}_n) = \xi_n + \xi_{n+1} = \xi_n \neq X_n$).
   d) $X$ is not a martingale (actually, $X$ is an increasing process, i.e. a very particular type of submartingale!).
   e) $X_n = \sum_{i=1}^n H_i \xi_i$ where $H_1 = 1$, $H_{n+1} = \xi_1 + \ldots + \xi_n$ is predictable and $|H_n| \leq n$ for all $n$, so $X$ is a martingale.

2. a) By Jensen’s inequality, we know that
   $$E(\varphi(X_{n+1})|\mathcal{F}_n) \geq \varphi(E(X_{n+1}|\mathcal{F}_n)).$$
   We also know that $E(X_{n+1}|\mathcal{F}_n) \geq X_n$, since $(X_n)$ is a submartingale. The function $\varphi$ needs therefore to be also increasing in order to ensure that
   $$E(\varphi(X_{n+1})|\mathcal{F}_n) \geq \varphi(X_n).$$
   b) In particular, $(X_n^2)$ need not necessarily be a submartingale, whereas $(\exp(X_n))$ is.

3. a) $T$ is not a stopping time, but it is bounded.
   b) $T$ is an unbounded stopping time (notice that actually, $T = \inf\{n \geq 1 : S_n \geq 0\}$).
   c) $T$ is not a stopping time, but it is bounded.
   d) $T$ is a bounded stopping time.
4. a) Yes. Let us denote by $(\xi_n, n \geq 1)$ the process defined as $\xi_n = +1$ if the $n^{th}$ coin toss falls on “heads” and $\xi_n = -1$ if the $n^{th}$ coin toss falls on “tails”. The bet $H_n$ at time $n$ is given by

$$H_1 = 1, \quad H_{n+1} = \begin{cases} 0, & \text{if } \xi_n = +1, \\ 2H_n, & \text{if } \xi_n = -1, \end{cases}$$

and the gain of the player at time $n$ is given by

$$G_n = \sum_{i=1}^{n} H_i \xi_i.$$ 

This process can indeed be shown to be a martingale. Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)$. Then observe that $(\mathcal{F}_n, n \in \mathbb{N})$ is the natural filtration of the process $(G_n, n \in \mathbb{N})$. Indeed, the knowledge of the random variables $G_1, \ldots, G_n$ is equivalent to that of the random variables $\xi_1, \ldots, \xi_n$. So $G_n$ is $\mathcal{F}_n$-measurable, $\mathbb{E}(|G_n|) < \infty$ for all $n$ (this requires a quick and easy check) and

$$\mathbb{E}(G_{n+1}|\mathcal{F}_n) = \mathbb{E}(G_n + H_{n+1} \xi_{n+1}|\mathcal{F}_n) = \mathbb{E}(G_n|\mathcal{F}_n) + \mathbb{E}(H_{n+1} \xi_{n+1}|\mathcal{F}_n)$$

$$\overset{(*)}{=} G_n + H_{n+1} \mathbb{E}(\xi_{n+1}|\mathcal{F}_n) = G_n + H_{n+1} \mathbb{E}(\xi_{n+1}) = G_n + 0 = G_n,$$

where $(*)$ comes from the fact that both $G_n$ and $H_{n+1}$ are $\mathcal{F}_n$-measurable.

b) Suppose that the first “heads” comes out at time $T = n \geq 1$. Then the gain of the player is

$$-1 - 2 - 4 - \ldots - 2^{n-1} + 2^n = 1,$$

which does not depend on $n$. So whatever the time $T$, the gain of the player at this time is 1 franc.

c) There is a seeming contradiction, as $G_0 = 0$ and $G_T = 1$, so $\mathbb{E}(G_T) = 1 \neq 0 = \mathbb{E}(G_0)$. But as we will see, stopping a martingale at a random unbounded time is not innocent...