Matlab Solutions 1

a) Recall that $\int_{\mathbb{R}} e^{-y^2} \, dy = \Gamma(1/2) = \sqrt{\pi}$ (to see it, square the LHS, switch to polar coordinates $(r, \theta)$ and use Fubini, to integrate $re^{-r^2}$). Using substitution $y^2 = \frac{(x-\mu)^2}{2\sigma^2}$, i.e. $y = \frac{x-\mu}{\sqrt{2}\sigma}$, we have

$$\sqrt{\pi} = \int_{\mathbb{R}} e^{-y^2} \, dy = \int_{\mathbb{R}} \frac{1}{\sqrt{2\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \, dx.$$ 

Hence

$$\int_{\mathbb{R}} p_X(x) \, dx = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \, dx = 1.$$ 

Now, let's compute $E(X)$:

$$E(X) = \int_{\mathbb{R}} x p_X(x) \, dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} x \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \, dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} (x-\mu) \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \, dx + \mu$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} u \exp\left(-\frac{u^2}{2\sigma^2}\right) \, du + \mu$$

where the last equality holds because the function $u \mapsto f(u) := \exp\left(-\frac{u^2}{2\sigma^2}\right)$, which is an odd function (i.e. $f(-u) = -f(u)$), is integrated on a symmetric domain ($\mathbb{R}$), centered on 0.

We can use this computation to compute Var($X$):

$$\text{Var}(X) = E(X^2) - E(X)^2 = \int_{\mathbb{R}} x^2 p_X(x) \, dx - \mu^2$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} x^2 \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \, dx - \mu^2$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\sqrt{2\sigma^2} y + \mu\right)^2 e^{-y^2} \, dy - \mu^2$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{\mathbb{R}} y^2 e^{-y^2} \, dy$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{\mathbb{R}} y \, dy \left(-\frac{1}{2} e^{-y^2}\right) \, dy = \frac{2\sigma^2}{\sqrt{\pi}} \int_{\mathbb{R}} \frac{1}{2} e^{-y^2} \, dy$$

$$= \sigma^2,$$

where the next to last equality holds because of integration by parts.

Now,

$$E((X - \mu)^3) = \int_{\mathbb{R}} (x - \mu)^3 p_X(x) \, dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} (x - \mu)^3 \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \, dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} u^3 \exp\left(-\frac{u^2}{2\sigma^2}\right) \, du$$

$$= 0,$$
because the integrand is odd on $\mathbb{R}$, the domain of integration, which is symmetric with respect to 0.

Finally,

\[
\mathbb{E}((X - \mu)^4) = \int_{\mathbb{R}} (x - \mu)^4 p_X(x) \, dx \\
= \frac{4\sigma^4}{\sqrt{\pi}} \int_{\mathbb{R}} y^4 e^{-y^2} \, dy = \frac{8\sigma^4}{\sqrt{\pi}} \int_0^{+\infty} y^4 e^{-y^2} \, dy \\
= \frac{4\sigma^4}{\sqrt{\pi}} \int_0^{+\infty} t^{3/2} e^{-t} \, dt \\
= \frac{4\sigma^4}{\sqrt{\pi}} \cdot \Gamma(5/2) = \frac{4\sigma^4}{\sqrt{\pi}} \cdot \Gamma(1/2) \cdot \frac{3}{4} \\
= 3\sigma^4.
\]

**b1)** See Matlab code.

**b2)** The more the skewness is positive, the more the sample is right-skewed. The more it’s negative, the more the sample is left-skewed. It is important to understand that the skewness of the sample might not be a good approximation of the real skewness coefficient of the original distribution (to answer such a question, one would need to study statistical theory).

There are various interpretations to kurtosis. One of them (the classical, which holds only for unimodal typically non-skewed distributions) is that if the kurtosis is larger than 3, one could say it is peaked around its mean, with fat tails; and if it’s smaller than 3, its peak is lower and wider, with thinner tails.

**b3)** It’s obvious that the kurtosis is strictly positive (the variance is strictly positive). In fact, it’s easy to see that it is not smaller than 1 (use Cauchy-Schwarz or Bernoulli inequality). This is a general result: the kurtosis is always bounded below by $S^2 + 1$ (recall that $S$ is the skewness), hence in particular by 1.

**c)** If $X_1, X_2$ are two i.i.d. $\mathcal{N}(0, 1)$ random variables and $(X_1, X_2) = (R \cos \Theta, R \sin \Theta)$, then

\[
\frac{1}{2\pi} e^{-x^2/2} e^{-y^2/2} \, dx \, dy = \frac{1}{2\pi} re^{-r^2/2} \, dr \, d\theta = \left( \frac{1}{2\pi} \, d\theta \right) \left( re^{-r^2/2} \, dr \right) = \left( \frac{1}{2\pi} \, d\theta \right) \left( \frac{1}{2} e^{-t/2} \, dt \right),
\]

where the last equality holds thanks to the change of variable $t = r^2$. We notice that $R$ and $\Theta$ are independent (their joint density is the product of their respective marginal densities), and that $R$ has an exponential distribution (parameter 1/2) meanwhile $\Theta$ is uniformly distributed on $[0, 2\pi]$.

For the numerical computation, see Matlab code.

**d1)** Notice that the result is actually true for any two independent random variables, as in all generality we have

\[
\text{Cov}(X_1, X_2) = \mathbb{E}(X_1 X_2) - \mu_1 \mu_2 = \mathbb{E}(X_1)\mathbb{E}(X_2) - \mu_1 \mu_2 = 0.
\]

**d2)** See Matlab code.
d3) The CDF (cumulative distribution function) of $Y_2$ is

$$F_{Y_2}(x) = \mathbb{P}\{Y_2 \leq x\} = \mathbb{P}\{ZY_1 \leq x\}$$

$$= \mathbb{P}\{Y_1 \leq x | Z = +1\} \mathbb{P}\{Z= +1\} + \mathbb{P}\{-Y_1 \leq x | Z = -1\} \mathbb{P}\{Z= -1\}$$

$$= \frac{1}{2} \mathbb{P}\{Y_1 \leq x\} + \frac{1}{2} \mathbb{P}\{-Y_1 \leq x\} = \frac{1}{2} \mathbb{P}\{Y_1 \leq x\} + \frac{1}{2} \mathbb{P}\{Y_1 \geq -x\}$$

$$= \frac{1}{2} \left( \mathbb{P}\{Y_1 \leq x\} + \int_{-x}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \right)$$

$$= \frac{1}{2} \left( \mathbb{P}\{Y_1 \leq x\} - \int_{x}^{-\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du \right)$$

$$= \frac{1}{2} \left( \mathbb{P}\{Y_1 \leq x\} + \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du \right)$$

$$= \frac{1}{2} \mathbb{P}\{Y_1 \leq x\} + \frac{1}{2} \mathbb{P}\{Y_1 \leq x\}$$

$$= \mathbb{P}\{Y_1 \leq x\} = F_{Y_1}(x),$$

where we used the change of variable $u = -y$. Hence $Y_2 \sim \mathcal{N}(0,1)$.

Then, we use independence of $Z$ with respect to $Y_1$ to compute

$$\text{Cov}(Y_1, Y_2) = \text{Cov}(Y_1, ZY_1) = \mathbb{E}(Y_1 ZY_1) - \mathbb{E}(Y_1)\mathbb{E}(ZY_1) = \mathbb{E}(Z) \left( \mathbb{E}(Y_1^2) - \mathbb{E}(Y_1)^2 \right) = 0,$$

because $\mathbb{E}(Z)$ is zero.

d4) See Matlab code.

d5) When $Y_1$ takes some value $y \in \mathbb{R}$, the value of $Y_2$ is precisely $y$ up to its sign. Put differently, we have

$$|Y_2| = |Y_1|,$$

so $Y_1$ and $Y_2$ cannot be independent.

e1) We have

$$\text{Cov}(X_1, X_2) = \Sigma_{12} = 0 \iff \Sigma \text{ is a diagonal matrix}$$

$$\iff \forall x_1, x_2 \in \mathbb{R}, \ p_{X_1,X_2}(x_1, x_2) = p_{X_1}(x_1)p_{X_2}(x_2)$$

$$\iff X_1 \text{ and } X_2 \text{ are independent}.$$

e2) No (if they were, they would be independent but part d5) proves they can’t be).