Midterm Exam

SURNAME: .......................... FIRST NAME: .......................... SECTION: .............

!!! NO JUSTIFICATIONS REQUIRED IN THE QUIZ BELOW !!!
Simply tick the boxes (multiple correct answers are possible)

Exercise 1. Quiz. [box ticked correctly = +1 point, box ticked incorrectly = $-\frac{1}{2}$ point]

a) Let $\Omega = \{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ and $\mathbb{P}(\{\omega\}) = \frac{1}{16}$ for all $\omega \in \Omega$.

On the above probability space, we define moreover the following two random variables:

$X(\omega) = \omega_1 + \omega_2$, $Y(\omega) = \omega_2 - \omega_1$, as well as the following sub-$\sigma$-field of $\mathcal{F}$:

$\mathcal{G} = \sigma\{\{i\} \times \{1, 2, 3, 4\}, i = 1, \ldots, 4\}$.

Which of the following statements are correct?

- $X + Y$ is $\sigma(X, Y)$-measurable
- $X + Y$ is $\mathcal{G}$-measurable
- $\text{Cov}(X, Y) = 0$
- $X$ is $\sigma(X + Y)$-measurable
- $X - Y$ is $\mathcal{G}$-measurable
- $X$ and $Y$ are independent

b) Let $X_1, X_2$ be i.i.d. $\mathcal{N}(0, 1)$ random variables, and $Z$ be another random variable, independent of both $X_1$ and $X_2$, such that $\mathbb{P}(\{Z = +1\}) = \mathbb{P}(\{Z = -1\}) = \frac{1}{2}$. Which of the following random variables are Gaussian?

- $X_1 + ZX_2$
- $X_1 + Z (X_2 - X_1)$
- $(X_1 + X_2) + Z (X_1 + X_2)$
- $(X_1 + X_2) + Z (X_1 - X_2)$

On the above probability space, we define moreover the following two random variables:

- $\mathcal{F}$
- $\mathcal{G}$

Which of the following functions $F : \mathbb{R} \rightarrow \mathbb{R}$ are cdfs?

- $F(t) = \exp(-e^{-t})$, $t \in \mathbb{R}$
- $F(t) = \frac{t}{1 + |t|}$, $t \in \mathbb{R}$
- $F(t) = \begin{cases} 1 - \frac{1}{t}, & t \geq 1 \\ 0, & t < 1 \end{cases}$
- $F(t) = \frac{e^t}{1 + e^t}$, $t \in \mathbb{R}$

Which of the following functions $\phi : \mathbb{R} \rightarrow \mathbb{C}$ are characteristic functions?

- $\phi(t) \equiv 1$, $\forall t \in \mathbb{R}$
- $\phi(t) = e^{it}$, $t \in \mathbb{R}$
- $\phi(t) = \exp(-|t|)$, $t \in \mathbb{R}$
- $\phi(t) = \begin{cases} 1, & |t| \leq 1 \\ 0, & |t| > 1 \end{cases}$

BONUS e) [1 point] Let $c_1, \ldots, c_n \in \mathbb{R}$ and $x_1, \ldots, x_n \in \mathbb{R}$. The function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ defined as $\phi(t) = \sum_{j=1}^n c_j e^{itx_j}$ is a characteristic function (only one possible correct answer):

- for all values of $c_1, \ldots, c_n \in \mathbb{R}$ and $x_1, \ldots, x_n \in \mathbb{R}$,
- only if $x_j \geq 0$ for all $j \in \{1, \ldots, n\}$ and $\sum_{j=1}^n x_j = 1$,
- only if $c_j \geq 0$ for all $j \in \{1, \ldots, n\}$ and $\sum_{j=1}^n c_j = 1$,
- only if $c_1, \ldots, c_n \in \mathbb{R}$ and $x_1, \ldots, x_n \in \mathbb{R}$ are such that $\phi(t) \geq 0$ for all $t \in \mathbb{R}$.

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Exercise 2. [16 points] Let $\lambda > 0$ and $(X_n, n \geq 1)$ be a sequence of i.i.d. $\mathcal{E}(\lambda)$ random variables (i.e. $X_1$ is a non-negative random variable with cdf $F_{X_1}(t) = 1 - \exp(-\lambda t)$ for $t \geq 0$).

For $n \geq 1$, let also $Y_n = e^{X_n}$ and $S_n = \sum_{j=1}^{n} Y_j$.

a) Compute the pdf of $Y_1$.

b) For what values of $\lambda > 0$ does there exist a real number $\mu$ such that $\lim_{n \to \infty} \frac{S_n}{n} = \mu$ almost surely? Compute $\mu$ when it exists.

c) Using the central limit theorem, for what values of $\lambda > 0$ can you prove that the sequence of random variables

$$Z_n = \frac{S_n - n\mu}{\sqrt{n}}$$

converges in distribution to some limiting random variable $Z$ as $n \to \infty$? When it exists, what is the distribution of $Z$?

d) For what values of $\lambda > 0$ does there exist $s_0 > 0$ such that

$$\Lambda(s) = \log \mathbb{E}(e^{sY_1}) < +\infty \quad \forall |s| \leq s_0?$$

Exercise 3. [18 points] Let $(X_n, n \geq 1)$ be a sequence of i.i.d. $\mathcal{E}(1)$ random variables (see Ex. 2 for the definition). For $n \geq 1$, let also $R_n = \min(X_1, \ldots, X_n)$ and $T_n = \max(X_1, \ldots, X_n)$.

a1) Compute the cdf of $R_n$.

a2) Compute $\mathbb{E}(R_n)$.

a3) Does the sequence $n R_n$ converge in distribution to a limit as $n \to \infty$? If yes, compute the limiting cdf; if no, explain why.

BONUS: a4) [2 points] What can you say on $\mathbb{P}(\{\omega \in \Omega : \lim_{n \to \infty} n R_n(\omega) \text{ exists}\})$?

b1) Compute the cdf of $T_n$.

b2) Compute $\mathbb{E}(T_n)$.

b3) To what limit does the sequence $T_n - \log(n)$ converge in distribution as $n \to \infty$? Compute the limiting cdf. (NB: Here, “log” is the natural logarithm in base $e$.)

Hint: For this exercise, you may find the following facts useful:

- For a non-negative and continuous random variable $X$, $\mathbb{E}(X) = \int_{0}^{\infty} \mathbb{P}(|X > t|) \, dt$.

- For $0 < a < 1$ and $n \geq 1$, $\sum_{k=0}^{n-1} a^k = \frac{1 - a^n}{1 - a}$.

- For $x \in \mathbb{R}$, $e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$. 

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