Matlab Homework 4 (due Monday, December 14)

**Exercise 1.** Let \((\xi_n, n \geq 1)\) be a sequence of i.i.d. random variables such that \(P(\{\xi_1 = +1\}) = 1/3\) and \(P(\{\xi_1 = -1\}) = 2/3\), let \(S = (S_n, n \in \mathbb{N})\) be the process defined as \(S_0 = 0, S_n = \xi_1 + \ldots + \xi_n; n \geq 1\), and let \((\mathcal{F}_n, n \in \mathbb{N})\) be its natural filtration.

a) What type of process is \((S_n, n \in \mathbb{N})\)?

Let now \(T_b = \inf\{n \geq 0 : S_n \geq b\}\), where \(b \geq 1\) is an integer number. Notice that the stopping time \(T_b\) might take the value \(+\infty\) with positive probability; in this exercise, we are interested in estimating this probability. To this end, let \(X = (X_n, n \in \mathbb{N})\) be the process defined as \(X_n = 2^{S_n}, n \geq 0\).

b) Again, what type of process is \((X_n, n \in \mathbb{N})\)? Does this process converge almost surely to some limiting random variable \(X_\infty\) as \(n \to \infty\)? If yes, what is \(X_\infty\)? Does it hold also that \(E(X_\infty|\mathcal{F}_n) = X_n, \forall n \in \mathbb{N}\)?

c) Applying the optional stopping theorem to the process \(X\) (and justifying its use!), deduce what the value of \(P(\{T_b = +\infty\})\) is. Test this numerically!

**Exercise 2.** Let \((M_n, n \in \mathbb{N})\) be a square-integrable martingale with respect to some filtration \((\mathcal{F}_n, n \in \mathbb{N})\) and \((H_n, n \in \mathbb{N})\) be a predictable process with respect to \((\mathcal{F}_n, n \in \mathbb{N})\), such that \(|H_n(\omega)| \leq K_n\) for every \(\omega \in \Omega\) and \(n \in \mathbb{N}\).

Let also \((G_n, n \in \mathbb{N})\) be the process defined as \(G_0 = 0, G_n = \sum_{j=1}^{n} H_j (M_j - M_{j-1}), n \geq 1\). By the proposition seen in class, we know that \(G\) is a martingale.

a) Show that \(E(G_n^2) = \sum_{j=1}^{n} E \left( H_j^2 (A_j - A_{j-1}) \right) \), for every \(n \geq 1\), where \((A_n, n \in \mathbb{N})\) is the (unique) predictable and increasing process such that \((M_n^2 - A_n, n \in \mathbb{N})\) is a martingale.

b) Consider \(M = S\), the simple symmetric random walk. Find a sufficient condition on the process \(H\) (other than \(H \equiv 0\) :) such that there exists a random variable \(G_\infty\) with \(E(G_\infty|\mathcal{F}_n) = G_n, \forall n \in \mathbb{N}\).

c) Numerical application: still with \(M = S\) (i.e., \(M_n = S_n = \sum_{j=1}^{n} X_j\) with \(X_j\) i.i.d. \(\pm 1\) with equal probability), observe numerically how does the process \(G\) behave when \(n \to +\infty\) with the following \(H\)'s (which are all equal to 0 at time 0, by convention)

\[
H_n^{(1)} = \frac{1}{n} \quad H_n^{(2)} = \frac{X_{n-1}}{n} \quad H_n^{(3)} = \frac{X_{n-1}}{\sqrt{n}} \quad H_n^{(4)} = \frac{X_n}{\sqrt{n}} \quad H_n^{(5)} = \frac{\sum_{j=1}^{n-1} X_j}{n}
\]

NB: One of these \(H\)'s is problematic!

please turn the page %
Exercise 3. Let \((U_n, n \geq 1)\) be a sequence of i.i.d. random variables, all uniform on \([0, 1]\), and let \(\mathcal{F}_0 = \{\emptyset, \Omega\}, \mathcal{F}_n = \sigma(U_1, \ldots, U_n), n \geq 1\). Let us also define the three processes

\[
X_0 = 1/2, \quad X_{n+1} = \begin{cases} \frac{1+X_n}{2}, & \text{if } U_{n+1} > X_n, \\ \frac{X_n}{2}, & \text{if } U_{n+1} \leq X_n, \end{cases}
\]

\[
Y_0 = 1/2, \quad Y_{n+1} = \begin{cases} \frac{1+Y_n}{2}, & \text{if } U_{n+1} \leq Y_n, \\ \frac{Y_n}{2}, & \text{if } U_{n+1} > Y_n, \end{cases}
\]

and

\[
Z_0 = 1/2, \quad Z_{n+1} = \begin{cases} \frac{1+Z_n}{2}, & \text{if } U_{n+1} \leq 1/2, \\ \frac{Z_n}{2}, & \text{if } U_{n+1} > 1/2. \end{cases}
\]

a) Are these three processes confined to some interval?

b) Compute \(E(X_{n+1}|\mathcal{F}_n), E(Y_{n+1}|\mathcal{F}_n)\) and \(E(Z_{n+1}|\mathcal{F}_n)\).

c) Which of the three processes is a martingale with respect to \((\mathcal{F}_n, n \geq 1)\)?

d) Is this martingale converging a.s. as \(n\) goes to infinity? To what limiting random variable? You may run a matlab simulation in order to answer this question.

e) Run a simulation to see what are the other two processes doing! Plot in particular (what you will see is quite interesting...):

e1) a trajectory of each process over one hundred time slots;

e2) the histogram of all possible values taken by each process over ten thousand time slots.