Exercise 1. Let $\lambda > 0$ be fixed. For a given $n \geq \lceil 1/\lambda \rceil$, let $X_1^{(n)}, \ldots, X_n^{(n)}$ be i.i.d. Bernoulli($\lambda/n$) random variables and let $S_n = X_1^{(n)} + \ldots + X_n^{(n)}$. Using Lévy’s continuity theorem, show that

$$S_n \xrightarrow{d} Z,$$

where $Z \sim \mathcal{P}(\lambda)$.

Exercise 2. (birthday problem)
Let $(X_n, n \geq 1)$ be a sequence of i.i.d. random variables, each uniform on $\{1, \ldots, N\}$. Let also

$$T_N = \min\{n \geq 1 : X_n = X_m \text{ for some } m < n\}$$

(notice that whatever happens, $T_N \in \{2, \ldots, N+1\}$). Show directly that

$$\mathbb{P}\left(\left\{ \frac{T_N}{\sqrt{N}} \leq t \right\}\right) \xrightarrow{N \to \infty} 1 - e^{-t^2/2}, \quad \forall t \geq 0.$$

NB: Approximations are allowed here!

Application: Use this to obtain a rough estimate of $\mathbb{P}\left(\{T_{365} \leq 22\}\right)$ (i.e. what is the probability that among 22 people, at least two of them have their birthday on the same day?)

Exercise 3. Let $(X_n, n \geq 1)$ be a sequence of i.i.d. $\mathcal{E}(\lambda)$ random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e. $X_1$ admits the following pdf:

$$p_{X_1}(x) = \begin{cases} \lambda \exp(-\lambda x), & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Let also $S_n = X_1 + \ldots + X_n$. Find a tight upper bound on

$$\mathbb{P}\left(\{S_n \geq nt\}\right), \quad \text{for } t > \mathbb{E}(X_1) = \frac{1}{\lambda}.$$

Exercise 4. Let $(X_n, n \geq 1)$ be a sequence of i.i.d. $\mathcal{N}(0, 1)$ random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let also $S_n = X_1 + \ldots + X_n$. Find the exact value of

$$\mathbb{P}\left(\{S_n \geq nt\}\right), \quad \text{for } t > 0.$$
**Exercise 5.** For this exercise, you will need a generalization of the Cauchy-Schwartz inequality: Hölder’s inequality (written here in a slightly unusual form to help you with the exercise). This inequality says that if $X, Y$ are two integrable random variables, then for every $\alpha \in [0, 1],$

$$E(|X|^\alpha |Y|^{1-\alpha}) \leq E(|X|)^\alpha E(|Y|)^{1-\alpha}.$$ 

Preliminary: show that for $\alpha = 1/2$, this is nothing but Cauchy-Schwarz’ inequality.

Let now $X$ be a random variable such that $E(\exp(sX)) < \infty$ for every $s \in \mathbb{R}$.

a) Show that the function $\Lambda(s) = \log(E(\exp(sX)))$ is convex.

b) Show that the function $\Lambda^*(t) = \sup_{s \in \mathbb{R}} (st - \Lambda(s))$ is also convex.