Exercise 1. Check that the distributions below are well defined distributions and compute, when they exist, the mean and the variance of these distributions.

A) Discrete distributions:

a) Bernoulli $B(p), p \in [0, 1]: \Pr(X = 1) = p, \Pr(X = 0) = 1 - p$.

b) binomial $Bi(n, p), n \geq 1, p \in [0, 1]: \Pr(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \ 0 \leq k \leq n$.

c) Poisson $P(\lambda), \lambda > 0: \Pr(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}, \ k \geq 0$.

B) Continuous distributions:

d) uniform $U([a, b]), a < b: p_X(x) = \frac{1}{b-a} 1_{[a, b]}(x), \ x \in \mathbb{R}$.

e) Gaussian $N(\mu, \sigma^2), \mu \in \mathbb{R}, \sigma > 0: p_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \ x \in \mathbb{R}$.

e) Cauchy $C(\lambda), \lambda > 0: p_X(x) = \frac{1}{\pi} \frac{\lambda}{x^2 + \lambda^2}, \ x \in \mathbb{R}$.

Exercise 2. Let $X$ be a centered Gaussian random variable of variance $\sigma^2$. Compute:

a) $E(X^4)$.

b) $E(\exp(X))$.

c) $E(\exp(-X^2))$.

Exercise 3. (reverse Chebychev’s inequality)
Let $X$ be a square-integrable random variable such that $X \geq 0$ a.s. Let also $0 \leq t < E(X)$.

a) Show that

$$\Pr(\{X > t\}) \geq \frac{(E(X) - t)^2}{E(X^2)}.$$

Hint: Use Cauchy-Schwarz’s inequality.

b) Application: Check that the above inequality holds in the particular case $X \sim P(\lambda)$ and $t = 0$.
Exercise 4. Let $X$ be a centered random variable with variance $\sigma^2$. Using Chebychev’s inequality, show that:

a) $\mathbb{P}(\{|X| \geq a\}) \leq \frac{\sigma^2}{a^2}$ and $\mathbb{P}(\{|X| \geq a\}) \leq \frac{2\sigma^2}{a^2 + \sigma^2}$.

b) $\mathbb{P}(\{X \geq a\}) \leq \frac{\sigma^2}{a^2 + \sigma^2}$ (use $\psi(x) = (x + b)^2$ with $b \geq 0$, then minimize over $b$).