Exercise 1. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Using only the axioms given in the definition of a probability measure, show the following properties:

a) $P(A) \leq P(B)$, if $A \subset B$, $A, B \in \mathcal{F}$.

b) $P(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} P(A_n)$, if $A_n \in \mathcal{F}$, $\forall n \geq 1$.

c) $P(\bigcap_{n=1}^{\infty} A_n) = 0$, if $A_n \in \mathcal{F}$ and $P(A_n) = 0$, $\forall n \geq 1$.

d) $P(B \setminus A) = P(B) - P(A)$, if $A \subset B$, $A, B \in \mathcal{F}$.

e) $P(A^c) = 1 - P(A)$, if $A \in \mathcal{F}$.

f) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, if $A, B \in \mathcal{F}$.

as well as the following two properties, also known as “the continuity axioms”:

g) $P(\bigcup_{n \geq 1} A_n) = \lim_{n \to \infty} P(A_n)$, if $A_n \in \mathcal{F}$ and $A_n \subset A_{n+1}$, $\forall n \geq 1$.

h) $P(\bigcap_{n \geq 1} A_n) = \lim_{n \to \infty} P(A_n)$, if $A_n \in \mathcal{F}$ and $A_{n+1} \subset A_n$, $\forall n \geq 1$.

Exercise 2. One considers the following simplified roulette game:

\[
\begin{array}{cc}
1 & 2 \\
3 & 4 \\
\end{array}
\]

a) Let us assume equal probabilities for all numbers.

Is the family of events “red”, “odd” and “1 or 2” independent?

Are these events 2-by-2 independent?

b) Consider the same question in the case where the roulette is biased as follows:

$P(\{1\}) = P(\{2\}) = 0.3$, $P(\{3\}) = P(\{4\}) = 0.2$.

Exercise 3. Let $X_1, X_2$ be independent and identically distributed (i.i.d.) random variables such that $P(\{X_i = +1\}) = P(\{X_i = -1\}) = 1/2$ for $i = 1, 2$. Let also $Y = X_1 + X_2$ and $Z = X_1 - X_2$.

a) Are $Y$ and $Z$ independent?

b) Same question with $X_1, X_2$ i.i.d. $\sim \mathcal{N}(0, 1)$ random variables.
Exercise 4. Let $\Omega = \mathbb{R}^2$ and let us define on $\Omega$: $\mathcal{F} = \sigma(\{B_1 \times B_2, \ B_1, B_2 \in \mathcal{B}(\mathbb{R})\})$. Note that $\mathcal{F}$ is nothing but $\mathcal{B}(\mathbb{R}^2)$.

Let us also define the random variables $X_1(\omega) = \omega_1$ and $X_2(\omega) = \omega_2$ for $\omega = (\omega_1, \omega_2) \in \Omega$ and let finally $\mu$ be a probability distribution on $\mathbb{R}$. Think e.g. of $\mu = \mathcal{N}(0,1)$:

$$\mu(B) = \int_B \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx, \quad B \in \mathcal{B}(\mathbb{R}).$$

We are considering below two different probability measures defined on $(\Omega, \mathcal{F})$; we only specify them on the “rectangles” $B_1 \times B_2$ (a general theorem guarantees that these probability measures can be extended uniquely to the whole $\sigma$-field $\mathcal{F}$ generated by the rectangles).

a) $\mathbb{P}^{(1)}(B_1 \times B_2) = \mu(B_1) \cdot \mu(B_2)$.

b) $\mathbb{P}^{(2)}(B_1 \times B_2) = \mu(B_1 \cap B_2)$

In each case, describe what is the relation between the random variables $X_1$ and $X_2$. 