Exercise 1. Let \((\xi_n, n \geq 1)\) be a sequence of i.i.d. random variables such that \(\mathbb{P}(\{\xi_1 = +1\}) = p, \) \(\mathbb{P}(\{\xi_1 = -1\}) = q\), where \(0 < p < 1\) and \(q = 1 - p\). Let also \(S_0 = 1\) and \(S_{n+1} = S_n + \xi_{n+1}\), for \(n \geq 0\). Let us finally define the process \((X_n, n \geq 0)\) as
\[
X_n = \left(\frac{q}{p}\right)^{S_n}, \quad n \geq 0.
\]
a) Depending on the value of \(p\), what type of process is \((X_n, n \geq 0)\)?
b) Let now \(T = \inf\{n \geq 0 : S_n = 0 \text{ or } S_n = N\}\). Use the optional stopping theorem (specifying which version may be used here and justifying it) to compute the probability that the process \(S\) hits the value \(N\) before it hits the value 0.

Exercise 2. Let \((S_n, n \in \mathbb{N})\) be the simple symmetric random walk on \(\mathbb{Z}\) and \((\mathcal{F}_n, n \in \mathbb{N})\) be its natural filtration.

a) Is the process \((S^4_n, n \in \mathbb{N})\) a submartingale with respect to \((\mathcal{F}_n, n \in \mathbb{N})\)? Justify your answer.
b) Is the process \((S^4_n - n, n \in \mathbb{N})\) a submartingale with respect to \((\mathcal{F}_n, n \in \mathbb{N})\)? Justify your answer.

Hint: Recall that \((x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\).
c) Show that \(\mathbb{E}(S^4_{n+1}) = \mathbb{E}(S^4_n) + 6n + 1\) and deduce the value of \(\mathbb{E}(S^4_n)\) by induction on \(n\).

Hint: Recall that \(\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}\).
d) Compute \(\lim_{n \to \infty} \frac{\mathbb{E}(S^4_n)}{n^2}\). Can you make a parallel with something you already know?

Exercise 3. Let \(Y = (Y_n, n \in \mathbb{N})\) be the process defined recursively as
\[
Y_0 = 1, \quad Y_{n+1} = \begin{cases} 
3Y_n, & \text{with probability 1/2,} \\
\frac{Y_n}{2}, & \text{with probability 1/2.}
\end{cases}
\]
a) Is the process \(Y\) a submartingale, supermartingale or martingale with respect to its natural filtration \((\mathcal{F}_n, n \in \mathbb{N})\)? Justify your answer.
b) Compute \(\mathbb{E}(Y_n)\) and \(\text{Var}(Y_n)\) recursively, for all \(n \geq 1\).
c) Is the process \(Y\) confined to some interval?
d) Does there exist a random variable \(Y_\infty\) such that \(Y_n \xrightarrow{n \to \infty} Y_\infty\) almost surely?
e) If it exists, what is the random variable \(Y_\infty\)?

Hint: In order to answer this question rigorously, consider the process \(Z\) defined as \(Z_n = \log(Y_n)\).
f) If \(Y_\infty\) exists, does it also hold that \(Y_n = \mathbb{E}(Y_\infty | \mathcal{F}_n)\)?