Threshold Saturation via Coupling
Ruediger Urbanke, EPFL
February 18th, 2010
~ 2 hours

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Modern Coding Theory
Cambridge University Press
LDPC Ensemble [Gal63]
LDPC Ensemble [Gal63]

variables

(3,
LDPC Ensemble [Gal63]

(3, 6)
Asymptotic Analysis

[Luby, Mitzenmacher, Shokrollahi, Spielman, Steman, 1997]
Asymptotic Analysis

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\[ x = \epsilon (1 - (1 - x)^5)^2 \]
EXIT value
(ten Brink)

\[ x = \epsilon (1 - (1 - x)^5)^2 \]
Some more Fixed Points

\[ x = \epsilon (1 - (1 - x)^5)^2 \]
MAP versus BP

[Measson, Montanari, Richardson, U., 2004]
Protographs
[Thorpe, Andrews, and Dolinar, 2004]
What is a protograph?
What is a protograph?

M cover with $M=11$
What is a protograph?

edge bundle
What is a protograph?

permutation
What is a protograph?

permutation

same for each edge bundle
Protographs
protograph
chain of protographs
chain of coupled protographs
Convolutional LDPC Codes

Feldstroem and Zigangirov, 1999
Engdahl and Zigangirov, 1999
Engdahl, Lentmaier, and Zigangirov, 1999
Lentmaier, Truhachev, and Zigangirov, 2001
Tanner, Sridhara, Sridhara, Fuja, and Costello, 2004
Sridhara, Lentmaier, Costello, and Zigangirov, 2004
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Lentmaier, Fettweis, Zigangirov, and Costello, 2009

[ variations on construction, and some analysis ]

Example: (3, 6) ensemble, BEC
$\epsilon_{BP}(3, 6) \approx 0.42944$;
$\epsilon_{BP}(3, 6, L) \approx 0.48815; \text{ (close to } 1/2) $

Why do we care? Why does this happen?
Convolutional LDPC Codes

Feldstroem and Zigangirov, 1999
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Why do we care? Why does this happen?
Rate Loss

\[
R(1, r = k1, L) = \frac{(2L + 1) - (2(L + \hat{1}) - 1)/k}{2L + 1} = \frac{k - 1}{k} - \frac{2\hat{1}}{k(2L + 1)}.
\]
(3, 6, L) ensemble; L increasing
Performance Gain

(3, 6, L) ensemble; L increasing
Performance Gain

(3, 6, L) ensemble; L increasing
Performance Gain

(3, 6, L) ensemble; L increasing

\[ \approx 0.48815 \]
How does this happen?
How does this happen?

constellation
Randomized Ensemble (l,r,w,L)
Density Evolution for \((l,r,w,L)\)

\[
x_i = \epsilon \left( 1 - \frac{1}{w} \sum_{j=0}^{w-1} \left( 1 - \frac{1}{w} \sum_{k=0}^{w-1} x_{i+j-k} \right)^{r-1} \right)^{1-r}
\]
Proof Outline

(1) Existence of FP
Proof Outline

(2) Construction of EXIT curve

Area under curve

\[ \approx R = 1 - \frac{1}{r} \]

\[ \epsilon^* \approx \epsilon^{\text{MAP}} (l, r) \]
(2) Construction of EXIT curve

Area under curve

\[ \approx R = 1 - \frac{1}{r} \]

\[ \epsilon^* \approx \epsilon_{\text{MAP}}(l, r) \]
Proof Outline

(3) Operational Interpretation of the EXIT curve

For $\epsilon < \epsilon^*$ BP converges to trivial FP
Existence of FP

$(x, \epsilon^*)$
Existence of FP

(1) Long tail \( O(L) \)
Existence of FP

(1) Long tail \( O(L) \)

(2) Sharp transition \( O(w) \)
Existence of FP

1. Long tail $O(L)$
2. Sharp transition $O(w)$
3. Flat part with value close to $x_{\text{stab}}(\epsilon^*)$ $O(L)$

$$x = \epsilon^* \left(1 - (1 - x)^5\right)^2$$
Existence of FP

(1) Long tail $O(L)$

(2) Sharp transition $O(w)$

(3) Flat part with value close to $x_{\text{stab}}(\epsilon^*) O(L)$

$$x = \epsilon^* (1 - (1 - x)^5)^2$$

(4) $\epsilon^\text{BP} < \epsilon^* < 1$
Existence of FP

\[ x_i = \epsilon \left( 1 - \frac{1}{w} \sum_{j=0}^{w-1} \left( 1 - \frac{1}{w} \sum_{k=0}^{w-1} x_{i+j-k} \right)^{r-1} \right)^{1-1} \]

\[ i \in [-L, L] \quad x_i = 0 \ \forall \ i < -L; \ x_i = x_0 \ \forall \ i > 0 \]
Existence of FP

Proof involves the use of Brouwer’s FP theorem
Construction of EXIT curve
Construction of EXIT curve

To show that $\epsilon^* \approx \epsilon^{\text{MAP}}$
Construction of EXIT curve

To show that $\epsilon^* \approx \epsilon_{\text{MAP}}$

Tool: Area Theorem
Construction of EXIT curve

To show that $\epsilon^* \approx \epsilon^{\text{MAP}}$

Tool: Area Theorem
Construction of EXIT curve
Construction of EXIT curve

\[ \alpha \in [0, 1] \]
Construction of EXIT curve

\[ \alpha \in [0, 1] \]

\[ (x(\alpha), \epsilon(\alpha)) \rightarrow (h(\alpha), \epsilon(\alpha)) \]
Construction of EXIT curve

\[ \alpha \in [0, 1] \]

\[ (x(\alpha), \epsilon(\alpha)) \rightarrow (h(\alpha), \epsilon(\alpha)) \]

\[ (x^*, \epsilon^*) \]
Construction of EXIT curve
Construction of EXIT curve

\( x_{\text{stab}}(\epsilon^*) \)
Construction of EXIT curve

\[ x_{\text{stab}}(\epsilon^*) \]
Construction of EXIT curve

\[ x_{\text{stab}}(\epsilon^*) \]
Construction of EXIT curve
Construction of EXIT curve

Phase 1 \( \alpha \in \left[ \frac{3}{4}, 1 \right] \)
Construction of EXIT curve

Phase 1 \( \alpha \in \left[ \frac{3}{4}, 1 \right] \)

Phase 2 \( \alpha \in \left[ \frac{1}{2}, \frac{3}{4} \right] \)
Construction of EXIT curve

<table>
<thead>
<tr>
<th>Phase</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Phase 1</td>
<td>$\alpha \in \left[\frac{3}{4}, 1\right]$</td>
</tr>
<tr>
<td>Phase 2</td>
<td>$\alpha \in \left[\frac{1}{2}, \frac{3}{4}\right]$</td>
</tr>
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<td>Phase 3</td>
<td>$\alpha \in \left[\frac{1}{4}, \frac{1}{2}\right]$</td>
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Construction of EXIT curve
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Construction of EXIT curve
Operational Interpretation
Operational Interpretation

BP with

\[ \epsilon < \epsilon^* \]
Operational Interpretation

BP with $\epsilon < \epsilon^*$

Constellation in Phase 4
“wiggles” do not vanish in $L$
What We Can Prove

*Theorem* [Kudekar, Richardson, and U., 2009]
Consider transmission over the BEC(ε) using random elements from the ensemble \((1, r, L, w)\) with \(1 \geq 3\). Let \(ε^{BP}(1, r, L, w)\) denote the BP threshold and let \(R(1, r, L, w)\) denote the design rate of this ensemble. Then, in the limit as \(M\) tends to infinity, and for \(w > w_0(1, r)\),

\[
ε^{MAP}(l, r, L, w) \leq ε^{MAP}(l, r) + \frac{w}{L(1-(1-x^{MAP})r^{-1})^{1}},
\]

\[
ε^{BP}(l, r, L, w) \geq \left(ε^{MAP}(l, r) - w^{-\frac{1}{8}} \frac{81^2 r}{(1-2^{-\frac{1}{r}})^2}(1 - 4w^{-\frac{1}{8}})^{r1}\right).
\]

In the limit as \(M, L\) and \(w\) (in that order) tend to infinity,

\[
\lim_{w \to \infty} \lim_{L \to \infty} R(1, r, L, w) = 1 - \frac{1}{r},
\]

\[
\lim_{w \to \infty} \lim_{L \to \infty} ε^{BP}(1, r, L, w) = \lim_{L \to \infty} ε^{MAP}(1, r, L, w) = ε^{MAP}(1, r).
\]

coupling of systems converts MAP threshold into BP threshold
Code Design Principle

For MAP:

+ easy to get good threshold
  (3, 6)  0.48816
  (4, 8)  0.49774  cap=0.5
  (5, 10) 0.49948

+ easy to avoid errorfloor
  (min distance grows linear in M)

- increased codelength (by factor L)
- number of iterations and scaling?
Generalizations?
BAWGN Channel

(3, 6) ensemble
BAWGN Channel

(3, 6) ensemble
BAWGN Channel
BAWGN Channel
BAWGN Channel

BAWGN Channel

Thursday, February 18, 2010
(3, 6, L) ensemble; L increasing
BAWGN Channel

[Kudekar, Measson, Richardson, and U., 2010]

(3, 6, L) ensemble; L increasing
BAWGN Channel

[Kudekar, Measson, Richardson, and U., 2010]

(3, 6, L) ensemble; L increasing

$\approx 0.4789$
Fully Connected Graph

[Hassani and Macris, 2010]

\[
H_N(s) = -\frac{J}{N} \sum_{\langle i,j \rangle} s_i s_j - h \sum_{i=1}^{N} s_i
\]
K-SAT

\[(\overline{x}_1 \lor \overline{x}_2 \lor \overline{x}_4) \land (x_1 \lor \overline{x}_2) \land (x_2 \lor x_4 \lor x_5) \land (x_1 \lor x_2 \lor \overline{x}_5) \land (x_1 \lor \overline{x}_3 \lor x_5)\]

[Mezard and Montanari, 2009]
3-SAT

\[ \alpha = \text{number of clauses per variable} \]

[Mezard and Montanari, 2009]
Select References

Feldstroem and Zigangirov, 1999
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[introduction]
[variations on construction, and some analysis]
[density evolution, BEC]
[general channels]
[protographs]
[pseudocodeword analysis]

arXiv:1001.1826 [pdf, other]
Threshold Saturation via Spatial Coupling: Why Convolutional LDPC Ensembles Perform so well over the BEC
Shrinivas Kudekar, Tom Richardson, Ruediger Urbanke
Comments: 28 pages, 11 figures
Subjects: Information Theory (cs.IT)