Exercice 1

1. One has to show that $\langle B_{x,y} B_{x',y'} \rangle = \delta_{x,x'} \delta_{y,y'}$. We show it explicitly for two cases:

$$\langle B_{00} | B_{00} \rangle = \frac{1}{2} (\langle 00 | + \langle 11 |)(\langle 00 | + \langle 11 |))$$

$$= \frac{1}{2} (\langle 00 | 00 \rangle + \langle 00 | 11 \rangle + \langle 11 | 00 \rangle + \langle 11 | 11 \rangle).$$

Now we have

$$\langle 00 | 00 \rangle = \langle 0 | 0 \rangle \langle 0 | 0 \rangle = 1, \langle 00 | 11 \rangle = \langle 0 | 1 \rangle \langle 0 | 1 \rangle = 0,$$

$$\langle 11 | 00 \rangle = \langle 1 | 0 \rangle \langle 1 | 0 \rangle = 0, \langle 11 | 11 \rangle = \langle 1 | 1 \rangle \langle 1 | 1 \rangle = 1.$$

Thus we get that $\langle B_{00} | B_{00} \rangle = \frac{1}{2} (1 + 0 + 0 + 1) = 1$. Now let us consider

$$\langle B_{00} | B_{01} \rangle = \frac{1}{2} (\langle 00 | + \langle 11 |)(\langle 01 | + \langle 10 |))$$

$$= \frac{1}{2} (\langle 00 | 01 \rangle + \langle 00 | 10 \rangle + \langle 11 | 01 \rangle + \langle 11 | 10 \rangle)$$

$$= \frac{1}{2} (0 + 0 + 0 + 0) = 0.$$

2. The proof is by contradiction. Suppose there exist $a_1, b_1$ and $a_2, b_2$ such that

$$|B_{00}\rangle = (a_1 |0 \rangle + b_1 |1 \rangle) \otimes (a_2 |0 \rangle + b_2 |1 \rangle).$$

Then we must have

$$\frac{1}{2} (\langle 00 | + \langle 11 |) = a_1 b_2 |00 \rangle + a_1 b_2 |01 \rangle + b_1 a_2 |10 \rangle + a_2 b_2 |11 \rangle.$$

Since the states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ form a basis one has

$$\frac{1}{2} = a_1 a_2, \frac{1}{2} = b_1 b_2, a_1 b_2 = 0, b_1 a_2 = 0.$$

The third equality indicates that either $a_1 = 0$ or $b_2 = 0$ (or both). If $a_1 = 0$ we get a contradiction with the first equation. If on the other hand $b_2 = 0$, we get a contradiction with the second one. Therefore, there does not exist $|\psi_1\rangle$ and $|\psi_2\rangle$ such that $|B_{00}\rangle$ can be written as $|\psi_1\rangle \otimes |\psi_2\rangle$. Therefore, $B_{00}$ is entangled.
3. We have

\[ |\gamma\rangle \otimes |\gamma\rangle = (\cos(\gamma) |0\rangle + \sin(\gamma) |1\rangle) \otimes (\cos(\gamma) |0\rangle + \sin(\gamma) |1\rangle) \]

\[ = \cos^2(\gamma) |00\rangle + \cos(\gamma) \sin(\gamma) |01\rangle + \sin(\gamma) \cos(\gamma) |10\rangle + \sin^2(\gamma) |11\rangle. \]

Similarly,

\[ |\gamma\perp\rangle \otimes |\gamma\perp\rangle = \cos^2(\gamma\perp) |00\rangle + \cos(\gamma\perp) \sin(\gamma\perp) |01\rangle + \sin(\gamma\perp) \cos(\gamma\perp) |10\rangle + \sin^2(\gamma\perp) |11\rangle. \]

A picture shows that \( \cos(\gamma\perp) = -\sin(\gamma) \) and \( \sin(\gamma\perp) = \cos(\gamma) \) (this also allows to check that \( \langle \gamma| \gamma\perp \rangle = 0 \)). Therefore, \( \cos^2(\gamma\perp) = \sin^2(\gamma) \), \( \sin^2(\gamma\perp) = \cos^2(\gamma) \) and \( \cos(\gamma\perp) \sin(\gamma\perp) = -\cos(\gamma) \sin(\gamma) \). We find that

\[ |\gamma\rangle \otimes |\gamma\rangle + |\gamma\perp\rangle \otimes |\gamma\perp\rangle = (\cos^2(\gamma) + \sin^2(\gamma)) |00\rangle + (\sin^2(\gamma) + \cos^2(\gamma)) |11\rangle, \]

and the terms \( |01\rangle \) and \( |10\rangle \) cancel. Finally,

\[ \frac{1}{\sqrt{2}} (|\gamma\rangle \otimes |\gamma\rangle + |\gamma\perp\rangle \otimes |\gamma\perp\rangle) = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) = |B_{00}\rangle. \]

4. From the rule for the tensor product

\[
\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix},
\]

we get for the basis states

\[ |0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \]

\[ |1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \]
Thus,

\[ |B_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\
0 \\
0 \\
1 \end{pmatrix}, \]

\[ |B_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\
1 \\
1 \\
0 \end{pmatrix}, \]

\[ |B_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\
0 \\
0 \\
-1 \end{pmatrix}, \]

\[ |B_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\
1 \\
-1 \\
0 \end{pmatrix}. \]

**Exercice 2**

1. By definition of the tensor product:

\[(H \otimes I) |x\rangle \otimes |y\rangle = H |x\rangle \otimes I |y\rangle = H |x\rangle \otimes |y\rangle.\]

Also, one can use that

\[ H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\
1 & -1 \end{pmatrix} \]

to show that always

\[ H |x\rangle = \frac{1}{\sqrt{2}} (|0\rangle + (-1)^x |1\rangle). \]

Thus,

\[(H \otimes I) |x\rangle \otimes |y\rangle = \frac{1}{\sqrt{2}} (|0\rangle \otimes |y\rangle + (-1)^x |1\rangle \otimes |y\rangle). \]

Note that this state is not entangled. Indeed \((H \otimes I) |x\rangle \otimes |y\rangle = H |x\rangle \otimes |y\rangle\) which is a tensor product state.

Now we apply ‘\(CNOT\)’. By linearity, we can apply it to each term separately. Thus,

\[(CNOT)(H \otimes I) |x\rangle \otimes |y\rangle = \frac{1}{\sqrt{2}} ((CNOT) |0\rangle \otimes |y\rangle + (-1)^x(CNOT) |1\rangle \otimes |y\rangle)\]

\[= \frac{1}{\sqrt{2}}(|0\rangle \otimes |y\rangle + (-1)^x |1\rangle \otimes |y \oplus 1\rangle)\]

\[= |B_{xy}\rangle. \]

2. Let us first start with \(H \otimes I\). We use the rule

\[ \begin{pmatrix} a & b \\
c & d \end{pmatrix} \otimes \begin{pmatrix} e & f \\
g & h \end{pmatrix} = \begin{pmatrix} ae & af & be & bf \\
ag & ah & bg & bh \\
ce & cf & de & df \\
ge & ch & dg & dh \end{pmatrix}, \]

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Thus we have
\[
\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.
\]

For \((\text{CNOT})\), we use the definition:
\[
(\text{CNOT}) |x\rangle \otimes |y\rangle = |x\rangle \otimes |y \oplus x\rangle,
\]
which implies that the matrix elements are
\[
\langle x'y'| CNOT |xy\rangle = \langle x', y'| x, y \otimes x \rangle = \langle x'| x \rangle \langle y'| y \oplus x \rangle = \delta_{xx'}\delta_{y\oplus x,y'}.
\]
We obtain the following table with columns \(xy\) and rows \(x'y'\):

<table>
<thead>
<tr>
<th></th>
<th>00</th>
<th>01</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>00</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>01</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>11</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

For the matrix product \((\text{CNOT})(H \otimes I)\), we find that
\[
(\text{CNOT})H \otimes I = \frac{1}{\sqrt{2}} \begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} I & I \\ I & -I \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ X & -X \end{pmatrix},
\]
where \(X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\). Thus,
\[
(\text{CNOT})(H \otimes I) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}.
\]

One can check that for example \(|B_{00}\rangle = (\text{CNOT})(H \otimes I) |0\rangle \otimes |0\rangle\). Finally to check the unitarity, we have to check that \(UU^\dagger = U^\dagger U = I\) for \(U = H \otimes I\), \(\text{CNOT}\) and \((\text{CNOT})(H \otimes I)\). We leave this to the reader.

3. Let \(U = (\text{CNOT})(H \otimes I)\). We have
\[
|B_{xy}\rangle = U |x\rangle \otimes |y\rangle, \quad \langle B_{x'y'}| = \langle x'| \otimes \langle y'| U^\dagger.
\]
Thus,
\[
\langle B_{x'y'}| B_{xy}\rangle = \langle x'| \otimes \langle y'| U^\dagger U |x\rangle \otimes |y\rangle
= \langle x'| \otimes \langle y'| I |x\rangle \otimes |y\rangle
= \langle x'| x \rangle \langle y'| y \rangle = \delta_{xx'}\delta_{yy'}.
\]
Exercice 3

1. The possible outcomes of the measurement are simply the basis states. Let us compute the probability that the first basis state $|\alpha\rangle \otimes |\beta\rangle$ is the outcome. According to the measurement principle:

\[ \text{Prob}(\alpha, \beta) = \left| \langle \alpha | \otimes \langle \beta | \langle B_{00} \rangle \right|^2 \]

Using $|B_{00}\rangle = \frac{1}{\sqrt{2}} |\alpha\rangle \otimes |\alpha\rangle + \frac{1}{\sqrt{2}} |\alpha_{\perp}\rangle \otimes |\alpha_{\perp}\rangle$ (see exercise 1 and choose $\gamma = \alpha$) we get

\[ \text{Prob}(\alpha, \beta) = \frac{1}{2} |\langle \alpha | \alpha \rangle \langle \beta | \alpha \rangle + \langle \alpha | \alpha_{\perp} \rangle \langle \beta | \alpha_{\perp} \rangle|^2 \]
\[ = \frac{1}{2} (\cos(\alpha - \beta))^2 \]

For the three other probabilities we have

\[ \text{Prob}(\alpha, \beta_{\perp}) = \frac{1}{2} (\cos(\alpha - \beta_{\perp}))^2 = \frac{1}{2} (\sin(\alpha - \beta))^2 \]
\[ \text{Prob}(\alpha_{\perp}, \beta) = \frac{1}{2} (\cos(\alpha_{\perp} - \beta))^2 = \frac{1}{2} (\sin(\alpha - \beta))^2 \]
\[ \text{Prob}(\alpha_{\perp}, \beta_{\perp}) = \frac{1}{2} (\cos(\alpha_{\perp} - \beta_{\perp}))^2 = \frac{1}{2} (\cos(\alpha - \beta))^2 \]

2. In her lab Alice observes $|\alpha\rangle$ ou $|\alpha_{\perp}\rangle$. Using the results above (with $\cos^2 + \sin^2 = 1$) we find the probabilities

\[ \text{Prob}(\alpha) = \text{Prob}(\alpha, \beta) + \text{Prob}(\alpha, \beta_{\perp}) = \frac{1}{2} \]

et

\[ \text{Prob}(\alpha_{\perp}) = \text{Prob}(\alpha_{\perp}, \beta) + \text{Prob}(\alpha_{\perp}, \beta_{\perp}) = \frac{1}{2} \]

De même Bob dans son labo observe $|\beta\rangle$ ou $|\beta_{\perp}\rangle$ avec probabilités 1/2. So from the perspective of Alice and Bob each quantum bit is completely random!

3. First only Alice measures. The resulting states are calculated by acting with the projectors on $|B_{00}\rangle$:

\[ (|\alpha\rangle \langle \alpha | \otimes I) |B_{00}\rangle, \quad (|\alpha_{\perp}\rangle \langle \alpha_{\perp} | \otimes I) |B_{00}\rangle \]

Using the formula of exercise 1 for $\gamma = \alpha$ we find the two states

\[ \frac{1}{\sqrt{2}} |\alpha\rangle \otimes |\alpha\rangle, \quad \frac{1}{\sqrt{2}} |\alpha_{\perp}\rangle \otimes |\alpha_{\perp}\rangle \]

Since we should normalise the states we must discard the $1/\sqrt{2}$ in these formulas. The probabilities are

\[ \left| \langle \alpha | \otimes \langle \alpha | (|B_{00}\rangle) \right|^2 = \frac{1}{2}, \quad \left| \langle \alpha_{\perp} | \otimes \langle \alpha_{\perp} | (|B_{00}\rangle) \right|^2 = \frac{1}{2} \]
Now Bob measures. Thus the states just obtained after Alice’s measurement are projected with the projectors $I \otimes |\beta\rangle\langle\beta|$ or $I \otimes |\beta_\perp\rangle\langle\beta_\perp|$.

- If after Alice’s measurement the state is $|\alpha\rangle \otimes |\alpha\rangle$ (occurs with prob $1/2$) when Bob measures the state becomes (proportional to)

$$ (I \otimes |\beta\rangle\langle\beta|)(|\alpha\rangle \otimes |\alpha\rangle) = \langle \beta | \alpha \rangle |\alpha\rangle \otimes |\beta\rangle , \quad (I \otimes |\beta_\perp\rangle\langle\beta_\perp|)(|\alpha\rangle \otimes |\alpha\rangle) = \langle \beta_\perp | \alpha \rangle |\alpha\rangle \otimes |\beta_\perp\rangle $$

with probabilities

$$ \frac{1}{2} |\langle \alpha | \otimes \langle \beta | \alpha \rangle \otimes |\alpha\rangle|^2 = \frac{1}{2} (\cos(\alpha - \beta))^2 , \quad \frac{1}{2} |\langle \alpha | \otimes \langle \beta_\perp | \alpha \rangle \otimes |\alpha\rangle|^2 = \frac{1}{2} (\sin(\alpha - \beta))^2 $$

- If after Alice’s measurement the state is $|\alpha_\perp\rangle \otimes |\alpha_\perp\rangle$ (occurs with prob $1/2$) when Bob measures the state becomes (proportional to)

$$ (I \otimes |\beta\rangle\langle\beta|)(|\alpha_\perp\rangle \otimes |\alpha_\perp\rangle) = \langle \beta_\perp | \alpha_\perp \rangle |\alpha_\perp\rangle \otimes |\beta\rangle , \quad (I \otimes |\beta_\perp\rangle\langle\beta_\perp|)(|\alpha_\perp\rangle \otimes |\alpha_\perp\rangle) = \langle \beta_\perp | \alpha_\perp \rangle |\alpha_\perp\rangle \otimes |\beta_\perp\rangle $$

with probabilities

$$ \frac{1}{2} |\langle \alpha_\perp | \otimes \langle \beta_\perp | \alpha_\perp \rangle \otimes |\alpha_\perp\rangle|^2 = \frac{1}{2} (\sin(\alpha - \beta))^2 , \quad \frac{1}{2} |\langle \alpha_\perp | \otimes \langle \beta | \alpha_\perp \rangle \otimes |\alpha_\perp\rangle|^2 = \frac{1}{2} (\cos(\alpha - \beta))^2 $$

4. The previous question implies that when Alice does the measurement first and Bob after :

- Alice got the result $|\alpha\rangle$ or $|\alpha_\perp\rangle$ with prob $1/2$.
- Bob got in his lab the result $|\beta\rangle$ with probability

$$ \frac{1}{2} (\cos(\alpha - \beta))^2 + \frac{1}{2} (\sin(\alpha - \beta))^2 = \frac{1}{2} $$

or the result $|\beta_\perp\rangle$ with probability

$$ \frac{1}{2} (\sin(\alpha - \beta))^2 + \frac{1}{2} (\cos(\alpha - \beta))^2 = \frac{1}{2} $$

5. Summarising, this exercise has shown that the observations of Alice and Bob in each lab are the same whether the measurements are done simultaneously or in a series. With no communication between Alice and Bob the net result is :

- Alice chooses a measurement basis $\{|\alpha\rangle, |\alpha_\perp\rangle\}$ and gets the outcomes $|\alpha\rangle$ or $|\alpha_\perp\rangle$ with probability $1/2$;
- Bob chooses a measurement basis $\{|\beta\rangle, |\beta_\perp\rangle\}$ and gets the outcomes $|\beta\rangle$ or $|\beta_\perp\rangle$ with probability $1/2$.

With no communication the entanglement (intrication) is never detectable by local operations. Quantum bits in each separate lab appear to Alice and Bob as completely disordered or random.