**Exercise 1 Production of Bell states**

a) Direct computation gives

\[
(CNOT) (H \otimes I) |x\rangle \otimes |y\rangle = (CNOT) H |x\rangle \otimes |y\rangle \\
= (CNOT) \frac{1}{\sqrt{2}} (|0\rangle + (-1)^x |1\rangle) \otimes |y\rangle \\
= \frac{1}{\sqrt{2}} CNOT |0, y\rangle + \frac{(-1)^x}{\sqrt{2}} CNOT |1, y\rangle \\
= \frac{1}{\sqrt{2}} |0, y\rangle + \frac{(-1)^x}{\sqrt{2}} |1, y \oplus 1\rangle
\]

More explicitly, we enumerate all the cases :

\[
(CNOT) (H \otimes I) |00\rangle = (CNOT) \frac{1}{\sqrt{2}} (|00\rangle + |10\rangle) = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) = |B_{00}\rangle
\]
\[
(CNOT) (H \otimes I) |01\rangle = (CNOT) \frac{1}{\sqrt{2}} (|01\rangle + |11\rangle) = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle) = |B_{01}\rangle
\]
\[
(CNOT) (H \otimes I) |10\rangle = (CNOT) \frac{1}{\sqrt{2}} (|00\rangle - |10\rangle) = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle) = |B_{10}\rangle
\]
\[
(CNOT) (H \otimes I) |11\rangle = (CNOT) \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle) = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle) = |B_{11}\rangle
\]

b) The circuit corresponding to \( |B_{xy}\rangle = (CNOT) (H \otimes I) |x\rangle \otimes |y\rangle \) :

```
| x \rangle
| y \rangle  \xrightarrow{H}  \xrightarrow{\text{CNOT}}  | B_{xy} \rangle
```

c) The circuit corresponding to \( |x\rangle \otimes |y\rangle = (H \otimes I) (CNOT) |B_{xy}\rangle \) :

```
| B_{xy} \rangle  \rightarrow \xrightarrow{H}  | x \rangle \\
| y \rangle
```
Exercise 2 Construction of a multi-control- \( U \).

We show the quantum state at each stage of the circuit.

\[
\text{Input: } |c_1\rangle \otimes |c_2\rangle \otimes |c_3\rangle \otimes |0\rangle \otimes |0\rangle \otimes |c_t\rangle \\
\text{After the 1st Toffoli gate: } |c_1\rangle \otimes |c_2\rangle \otimes |c_3\rangle \otimes |c_1 \cdot c_2\rangle \otimes |0\rangle \otimes |c_t\rangle \\
\text{After the 2nd Toffoli gate: } |c_1\rangle \otimes |c_2\rangle \otimes |c_3\rangle \otimes |c_1 \cdot c_2 \cdot c_3\rangle \otimes |c_t\rangle \\
\text{After the controled-U gate: } |c_1\rangle \otimes |c_2\rangle \otimes |c_3\rangle \otimes |c_1 \cdot c_2 \cdot c_3\rangle \otimes U^{c_1,c_2,c_3} |c_t\rangle \\
\text{After the 3rd Toffoli gate: } |c_1\rangle \otimes |c_2\rangle \otimes |c_3\rangle \otimes |0\rangle \otimes |0\rangle \otimes U^{c_1,c_2,c_3} |c_t\rangle \\
\text{After the 4th Toffoli gate: } |c_1\rangle \otimes |c_2\rangle \otimes |c_3\rangle \otimes |0\rangle \otimes |0\rangle \otimes U^{c_1,c_2,c_3} |c_t\rangle \\
\]

Exercise 3 Construction of the Toffoli gate from a control-NOT (Indication : long calculation).

Note that \((CNOT) |x\rangle \otimes |y\rangle = |x\rangle \otimes |x \oplus y\rangle\) can also be represented as \(|x\rangle \otimes X^x |y\rangle\), where \(X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\) is one of the Pauli gates (see Chapter 3). Therefore, the circuit outputs the tensor product state \(|\psi\rangle\) given by

\[
|\psi\rangle = T |c_1\rangle \otimes SX^{c_1} T^\dagger X^{c_1} T^\dagger |c_2\rangle \otimes HTX^{c_1} T^\dagger X^{c_2} TX^{c_1} T^\dagger X^{c_2} H |t\rangle .
\]

We then verify explicitly all the cases of \(c_1\) and \(c_2\). The calculation largely uses the fact that all the quantum gates here are unitary (e.g., \(TT^\dagger = T^\dagger T = I\)); in particular, the gates \(X\) and \(H\) are involutory, i.e., \(X^2 = H^2 = I\).

For \(c_1 = 0\), we have

\[
|\psi\rangle = T |0\rangle \otimes ST^\dagger T^\dagger |c_2\rangle \otimes HTT^\dagger X^{c_2} TT^\dagger X^{c_2} H |t\rangle \\
= |0\rangle \otimes |c_2\rangle \otimes H (TT^\dagger) (X^{c_2} (TT^\dagger) X^{c_2}) H |t\rangle \\
= |0\rangle \otimes |c_2\rangle \otimes |t\rangle \\
= |0\rangle \otimes |c_2\rangle \otimes |t \oplus 0 \cdot c_2\rangle \\
\]

For \(c_1 = 1\) and \(c_2 = 0\), we calculate

\[
X T^\dagger X = \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & 1 \end{pmatrix} = e^{-i\pi/4} T
\]

and therefore we have

\[
|\psi\rangle = T |1\rangle \otimes SX T^\dagger X T^\dagger |0\rangle \otimes HTX T^\dagger TX T^\dagger H |t\rangle \\
= e^{i\pi/4} |1\rangle \otimes S (XT^\dagger X) T^\dagger |0\rangle \otimes H (T (X T^\dagger X) T^\dagger) H |t\rangle \\
= e^{i\pi/4} |1\rangle \otimes e^{-i\pi/4} ST T^\dagger |0\rangle \otimes |t\rangle \\
= e^{i\pi/4} |1\rangle \otimes e^{-i\pi/4} |0\rangle \otimes |t\rangle \\
= |1\rangle \otimes |0\rangle \otimes |t \oplus 1 \cdot 0\rangle .
\]
Finally, for $c_1 = c_2 = 1$, we calculate
\[
(TXT^\dagger X)^2 = \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} = e^{-i\pi/2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = e^{-i\pi/2} Z,
\]
and therefore we have
\[
HZH = X
\]
and therefore we have
\[
|\psi\rangle = T|1\rangle \otimes SXT^\dagger XT^\dagger |1\rangle \otimes HTXT^\dagger XH |t\rangle
\]
\[
= e^{i\pi/4} |1\rangle \otimes S \left(XT^\dagger X \right) T^\dagger |1\rangle \otimes H \left(TXT^\dagger X\right)^2 H |t\rangle
\]
\[
= e^{i\pi/4} |1\rangle \otimes e^{-i\pi/4} STT^\dagger |1\rangle \otimes e^{-i\pi/2} HZH |t\rangle
\]
\[
= e^{i\pi/4} |1\rangle \otimes e^{i\pi/4} |1\rangle \otimes e^{-i\pi/2} X |t\rangle
\]
\[
= |1\rangle \otimes |1\rangle \otimes X |t\rangle = |1\rangle \otimes |1\rangle \otimes |t \oplus 1\rangle.
\]

Exercise 4 Unitary representation of a reversible computation.

a) The relevant Hilbert space is $(\mathbb{C}^2)^{\otimes (n+1)}$. A unitary matrix is precisely a matrix that conserves the scalar product. Therefore, to show $U_f$ is a unitary matrix, we show it conserves the scalar product. For any two states $|x_1, \ldots, x_n; y\rangle$ and $|\hat{x}_1, \ldots, \hat{x}_n; \hat{y}\rangle$, we have
\[
\langle x_1, \ldots, x_n; y| U_f^\dagger U_f |\hat{x}_1, \ldots, \hat{x}_n; \hat{y}\rangle
\]
\[
= \langle x_1, \ldots, x_n; y \oplus f(x_1, \ldots, x_n)|\hat{x}_1, \ldots, \hat{x}_n; \hat{y} \oplus f(\hat{x}_1, \ldots, \hat{x}_n)\rangle
\]
\[
= \begin{cases} 
1 & \text{if } x_i = \hat{x}_i \text{ for all } i \text{ and } y = \hat{y} \\
0 & \text{otherwise},
\end{cases}
\]
which is equivalent to $\langle x_1, \ldots, x_n; y|\hat{x}_1, \ldots, \hat{x}_n; \hat{y}\rangle$.

b) When the output of $f$ is in $\{0, 1\}^m$, so we need $m$ storage bits $y_1, \ldots, y_m$. The relevant Hilbert space would be $(\mathbb{C}^2)^{\otimes (n+m)}$. The same calculation in (a) gives
\[
\langle x_1, \ldots, x_n; y_1, \ldots, y_m| U_f^\dagger U_f |\hat{x}_1, \ldots, \hat{x}_n; \hat{y}_1, \ldots, \hat{y}_m\rangle = \langle x_1, \ldots, x_n; y_1, \ldots, y_m|\hat{x}_1, \ldots, \hat{x}_n; \hat{y}_1, \ldots, \hat{y}_m\rangle.
\]
We conclude that $U_f$ is unitary.