Exercice 1 Refocusing

- One way is to find the corresponding matrices and multiply them together to show the identity. A probably simpler way is to show that for every basis vector both right hand side and left hand side operators give the same result. For example for $|\psi_0\rangle = |\uparrow\uparrow\rangle$, applying the matrices starting from the right hand side one, one obtains (using that $R_1$ flips a spin; check this!)

\[
|\psi_1\rangle = e^{-i\frac{J}{2} \frac{\pi}{2}} |\uparrow\uparrow\rangle = e^{-itJ} |\psi_0\rangle,
|\psi_2\rangle = (R_1 \otimes I) |\psi_1\rangle = e^{-itJ} |\downarrow\uparrow\rangle,
|\psi_3\rangle = e^{-i\frac{J}{2} \frac{\pi}{2}} |\psi_2\rangle = e^{-itJ} e^{-i\frac{J}{2} \frac{\pi}{2}} |\downarrow\uparrow\rangle = e^{-itJ} e^{itJ} |\downarrow\uparrow\rangle = |\downarrow\uparrow\rangle,
|\psi_4\rangle = (R_1 \otimes I) |\psi_3\rangle = (R_1 \otimes I) |\downarrow\uparrow\rangle = |\uparrow\uparrow\rangle,
\]

which shows that $|\psi_4\rangle = |\psi_0\rangle = (I_1 \otimes I_2) |\psi_0\rangle$. One can also check this for other basis vectors to see that the identity indeed holds.

- $J \ll 1$. Donc $\tau = \frac{\pi}{17} >> \pi$. Les $\pi$-pulses sont beaucoup plus rapides que l’évolution des spins nucléaires. L’idée est que en injectant deux $\pi$-pulses aux instants $\frac{\pi}{17}$ et $\tau$ on reforme l’état initial et donc tout se passe comme si les deux spins n’avaient pas évolué.

Exercice 2 Realization of the SWAP gate

- To find the matrix representation, it is sufficient to find how SWAP port operates on the basis vectors.

\[
\text{SWAP} |\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},
\text{SWAP} |\uparrow\downarrow\rangle = |\downarrow\uparrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},
\text{SWAP} |\downarrow\uparrow\rangle = |\uparrow\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},
\text{SWAP} |\downarrow\downarrow\rangle = |\downarrow\downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.
\]
Putting the resulting columns together we obtain the matrix representation

\[
\text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

Now it is easy to check that \((\text{SWAP})\text{(SWAP}^\dagger) = I\) which shows that \text{SWAP} is a unitary matrix.

- The Heisenberg Hamiltonian is obtained in the lecture notes and has the following matrix representation

\[
\mathcal{H} = \hbar J \vec{\sigma}_1 \cdot \vec{\sigma}_2 = \hbar J \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

To compute the evolution operator \(e^{-i\frac{\alpha H}{\hbar}}\), we notice that the matrix for \(\mathcal{H}\) has the diagonal representation

\[
\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix},
\]

where \(A = (1)\) and \(C = (1)\) are \(1 \times 1\) matrices and \(B = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}\) is a \(2 \times 2\) matrix. It is easy to show that for any complex number \(\alpha\)

\[
e^{\alpha \mathcal{H}} = \begin{pmatrix} e^{\alpha A} & 0 \\ 0 & e^{\alpha C} \end{pmatrix}.
\]

Thus it is sufficient to find these three matrix exponentials. \(A\) and \(C\) are numbers equal to 1, thus \(e^{\alpha A} = e^{\alpha C} = e^\alpha\).

Now it remains to find \(e^{\alpha B}\). Notice that we can write \(B = -I + 2X\) where \(X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\).

Notice that \(I\) and \(X\) commute with each other, i.e., \(IX = XI\). It is not difficult to show that the matrices that commute with each other can be treated like numbers while taking exponentials, namely, for any commuting matrix \(M, N\), \(e^{M+N} = e^M e^N\). (Notice that this formula is not in general correct). Therefore, we have

\[
e^{i\beta B} = e^{-i\beta I} e^{2i\beta X} = e^{-i\beta (I \cos(2\beta) + iX \sin(2\beta))},
\]

where we used the Euler’s formula for \(X\). It can be checked that at time \(t = \frac{\pi}{4}\), \(\alpha = -i\frac{\pi}{4}\), thus \(\beta = -\frac{\pi}{4}\). Hence, \(e^{\alpha A} = e^{\alpha B} = e^{-i\frac{\pi}{4}}\), and

\[
e^{i\beta B} = e^{i\frac{\pi}{4}} (\cos(\frac{\pi}{2})I - i \sin(\frac{\pi}{2}) X) = -ie^{i\frac{\pi}{4}} X = e^{-i\frac{\pi}{4}} X.
\]

Putting all together, the evolution operator at time \(t = \frac{\pi}{4}\) is

\[
e^{-i\frac{\pi}{4}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]

which neglecting the constant phase \(-i\frac{\pi}{4}\) is equal to the matrix for \text{SWAP}.
We can implement SWAP using three CNOT gates as depicted in Figure 1. To show this, one can simply check that starting from a general state $|x, y\rangle$, with $x, y \in \{0, 1\}$ after the first CNOT the resulting state is $|x, y \oplus x\rangle$, after the second CNOT the state is $|x \oplus (x \oplus y), x \oplus y\rangle = |x \oplus x \oplus y, x \oplus y\rangle = |y, x \oplus y\rangle$, where we used the identity $x \oplus x = 0$ for $x \in \{0, 1\}$. Finally after the third CNOT the state is $|y, (x \oplus y) \oplus y\rangle = |y, x\rangle$. Therefore the combination the three gates just swaps $x$ and $y$.

Note that this gives another proof that SWAP is a unitary matrix because it can be implemented as a combination of quantum gates and we know that all quantum gates are unitary.

![Diagram of SWAP gate using three CNOT gates](image)

**Figure 1** – Implementation of SWAP gate using three CNOT